FORTY-EIGHTH ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Calculus and Advanced Topics 2004

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Scoring formula: 4R - W + 40

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft head (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly 80 minutes to work.

Contributors to TMTA for the Annual Mathematics Contest:

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1. If f is a polynomial function whose 24th derivative has degree 3, what is the degree of the 15th derivative?

A) 0 B) 15 C) 45 D) 12 E) 9

2. If $\int_1^3 (ax^2 + 2x + c)dx = 20$ and $\int_{-1}^0 (ax^2 + 2x + c)dx = 5$, then 2a + c =

A) -4 B) 0 C) 2 D) 4 E) 6

3. Which function has an x value for which f is continuous, but the derivative f' does not exist?

A) $f(x) = x^2$ B) $f(x) = \sin x$ C) $f(x) = \tan x$ D) $f(x) = x^{\frac{2}{3}}$ E) $f(x) = e^{\frac{1}{x}}$

4. If f is a polynomial function of degree n, then f has

A) n critical numbers
B) n – 1 critical numbers
C) no more than n – 1 critical numbers
D) at least one critical number
E) n – 2 critical numbers

5. Find the rate of change of $f(x) = (x^2 - 4x + 3)(x - 2)$ with respect to x at the point (4,6).

A) 4 B) 8 C) 11 D) 3 E) 0

6. Evaluate the following limit. $\lim\limits_{x \to \infty} \frac{3x}{4x^2 - x}$

A) $\frac{3}{4}$ B) 0 C) -3 D) 3 E) Does not exist

7. The equation of the line tangent to the graph of $y = x^2 + x + 2$ at the point (1, 4) is

A) $y = 2x + 1$ B) $y = 3x + 1$ C) $y = 4x$ D) $y = 3$ E) $y = x + 2$

8. How many different values of r make the following function continuous on the entire real line?

$$f(x) = \begin{cases} 3x + 4, x > r \\ -x^2 \quad , x \leq r \end{cases}$$

A) 0 B) 1 C) 2 D) 3 E) 4

9. The value of $\sqrt{8+\sqrt{8+\sqrt{8+\ldots}}}$ is

A) 3 B) 4 C) $\frac{8121}{2500}$ D) $\frac{20\sqrt{8}}{17}$ E) $\frac{1}{2}+\frac{\sqrt{33}}{2}$

10. The following four statements and only these are found on a note card.

I. On this card exactly one statement is false.
II. On this card exactly two statements are false.
III. On this card exactly three statements are false.
IV. On this card exactly four statements are false.

Assuming each statement is true or false, the exact number of <u>false</u> statements among the four is

A) 0 B) 1 C) 2 D) 3 E) 4

11. Assume that the function f is continuous on [0, 5]. If $\int_1^2 f(x)dx = -4$ and $\int_1^3 f(x)dx = 7$, what is the value of $\int_3^2 f(x)dx$?

A) -11 B) 11 C) -3 D) 3 E) impossible to determine

12. The absolute minimum value for $f(x) = 3x^4 - 4x^3$ on $[-1, 2]$ is

A) 0 B) 7 C) 1 D) -1 E) does not exist

13. The average value of $f(x) = 3x^2 - 2x$ on [0, 4] is

A) 12 B) 10 C) 8 D) 40 E) 48

14. An antiderivative for the function defined by $f(x) = \dfrac{2}{1+e^{-x}}$ is

A) $\ln(1+e^x)$ B) $2\ln(e^{-x}+1)$ C) $2\ln(e^x+1)$ D) $\ln(1+e^{-x})$ E) $\frac{1}{2}\ln(e^x+1)$

15. If $\log(\log(\log(\log(x)))) = 0$ and log represents the base 10 logarithm, what is the value of x?

A) 10 B) 10^{100} C) $10^{10,000}$ D) $10^{10,000,000}$ E) $10^{10,000,000,000}$

16. A substance is 99% water. Some water evaporates, leaving a substance that is 98% water. How much of the water has evaporated?

A) An amount of water equivalent to 2% of the total original amount of the substance.
B) An amount of water equivalent to 1% of the total original amount of the substance.
C) An amount of water equivalent to 2.98% of the total original amount of the substance.
D) Half the original amount of water evaporated.
E) An amount of water equivalent to 50% of the total original amount of the substance.

17. Determine the points at which the graph of $f(x) = -x^4 + 3x^2 - 1$ has a horizontal tangent line.

A) $(0, -1)$, $\left(\frac{-\sqrt{3}}{2}, \frac{5}{4}\right)$, $\left(\frac{\sqrt{3}}{2}, \frac{5}{4}\right)$

D) $(0,1)$, $\left(\frac{-\sqrt{6}}{2}, \frac{5}{4}\right)$, $\left(\frac{\sqrt{6}}{2}, \frac{5}{4}\right)$

B) $(0, -1)$, $\left(\frac{-\sqrt{3}}{2}, \frac{-5}{4}\right)$, $\left(\frac{\sqrt{3}}{2}, \frac{5}{4}\right)$

E) $(0, -1)$, $\left(\frac{-\sqrt{6}}{2}, \frac{-5}{4}\right)$, $\left(\frac{\sqrt{6}}{2}, \frac{5}{4}\right)$

C) $(0, -1)$, $\left(\frac{-\sqrt{6}}{2}, \frac{5}{4}\right)$, $\left(\frac{\sqrt{6}}{2}, \frac{5}{4}\right)$

18. $f(x) = |x|$ at $x = 0$

A) is not continuous

C) has no tangent line

E) none of the above

B) has a vertical tangent line

D) has a horizontal tangent line

19. $\lim_{x \to 1} \frac{x-1}{x^3 - 1}$

A) equals 0 B) equals 1 C) does not exist D) equals $\frac{1}{3}$ E) equals $\frac{1}{4}$

20. What is $\frac{d^{2959}}{dx^{2959}}(-\sin x)$?

A) $\sin x$ B) $-\sin x$ C) $\cos x$ D) $-\cos x$ E) $-2959 \sin x$

21. Evaluate $\int_{-5}^{4} \frac{x^2}{|x|} dx$.

A) $\frac{41}{2}$ B) $\frac{9}{2}$ C) $\frac{41}{3}$ D) 9 E) 1

22. $\int x\sqrt{5-x}\, dx =$

A) $-\frac{1}{3}x^2(\sqrt{5-x})^3 + c$

C) $-5(5-x) + \frac{1}{2}(5-x)^2 + c$

B) $\frac{1}{3}x^2(\sqrt{5-x})^3 + c$

D) $\frac{2}{5}(\sqrt{5-x})^5 - \frac{10}{3}(\sqrt{5-x})^3 + c$

E) $-\frac{2}{5}(\sqrt{5-x})^5 + \frac{10}{3}(\sqrt{5-x})^3 + c$

23. A large roll of paper 8.5 inches wide is to be cut into individual sheets 11 inches long. A 500-sheet ream of this paper is 2.5 inches thick. The outside diameter of the roll is 36 inches and it is wrapped on a cardboard cylinder whose outside diameter is 6 inches. Approximately how many sheets of paper can be cut from this roll?

A) 9000 B) 15000 C) 18000 D) 200,000 E) 210,000

24. How many subsets of {n | n > 0 and n is a multiple of 3 less than 100} are also subsets of {n | n > 0 and n is a multiple of 4 less than 100}?

A) 64 B) 8 C) 128 D) 512 E) 256

25. If $\int f(x)\sin x dx = -f(x)\cos x + \int 3x^2 \cos x dx$, then $f(x)$ could be

A) $3x^2$ B) x^3 C) $-x^3$ D) $\sin x$ E) $\cos x$

26. Let f and g have continuous first and second derivatives everywhere. If $f(x) \le g(x)$ for all real x, which of the following must be true?

I. $f'(x) \le g'(x)$ for all real x
II. $f''(x) \le g''(x)$ for all real x
III. $\int_0^1 f(x)dx \le \int_0^1 g(x)dx$

A) none B) I only C) III only D) I and II only E) I, II and III

27. A rectangle of length $\frac{1}{4}\pi$ and height 4 is bisected by the x-axis and is in the first and fourth quadrants, with the leftmost edge on the y-axis. The graph of $y = \sin(x) + C$ divides the area of the rectangle in half. What is C?

A) $\frac{2\sqrt{2}-4}{\pi}$ B) $-\frac{1}{2}$ C) $-\frac{\pi}{4}$ D) $\frac{4-2\sqrt{2}}{\pi}$ E) $\frac{2\sqrt{2}-1}{\pi}$

28. What is the slope of the normal line to the graph of $y = x^3 - 3x^2 + 6x + 2004$ at its point of inflection?

A) -1 B) 1 C) 6 D) $-\frac{1}{3}$ E) $\frac{1}{3}$

29. Find an exponential function of the form $f(x) = a \cdot b^{-x} + c$ whose graph has horizontal asymptote y = 32 and passes through the point (2, 112).

A) $f(x) = 180(1.5)^x - 293$

B) $f(x) = 32(1.5)^{-x} + 32$

C) $f(x) = 180(1.5)^{-x} + 32$

D) $f(x) = 32x + 48$

E) $f(x) = 32(1.5)^x + 40$

30. Use $S = Pe^{rt}$ to find the number of years (accurate to one decimal place) needed for a continuously compounded investment of \$3000 to increase to \$4250 at an interest rate of 4%.

A) 0.9 years B) 8.7 years C) 26.1 years D) 2.6 years E) 1.4 years

31. Given that $a_3 = 5$, $a_5 = 8$ and $a_n + a_{n+1} + a_{n+2} = 7$ for all positive integers n, compute a_{2001}.

A) 5 B) 8 C) -6 D) 7 E) none of these

32. Of the following five statements, I to V, about the binary operation of averaging (arithmetic mean), those which are always true are

I. Averaging is associative.
II. Averaging is commutative
III. Averaging distributes over addition
IV. Addition distributes over averaging
V. Averaging has an identity.

A) II only B) I and II only C) II and III only D) II and IV only E) II and V only

33. Suppose $f(x) = e^{ax} + e^{bx}$. For how many different pairs (a,b) is $f'' - 2f' - 15f = 0$?

A) 0 B) 1 C) 2 D) 3 E) 4

34. Evaluate $\sum_{n=1}^{\infty} n(2^{1-n})$.

A) 2 B) 4 C) 6 D) 8 E) 10

35. Find k so that $f(x) = k(x-2)^2$ is a probability density function on [a, b].

A) $\dfrac{3}{(b-2)^3-(a-2)^3}$ C) $\dfrac{3}{(b-2)^3+(a-2)^3}$ E) $\dfrac{3}{(a-2)^3-(b-2)^3}$

B) $\dfrac{3}{b^3+a^3}$ D) $\dfrac{3}{b^3-a^3}$

36. Find $\lim_{h \to 0} \dfrac{\int_1^{1+h} \sqrt{x^5+8}\, dx}{h}$.

A) 0 B) 1 C) 3 D) $2\sqrt{2}$ E) Does not exist

37. For any real number $\boldsymbol{b}$, $\int_0^b |2x|\, dx$ is

A) $-b|b|$ B) b^2 C) $-b^2$ D) $b|b|$ E) None of these

38. Let $f(x) = \sqrt{x + \sqrt{0 + \sqrt{x + \sqrt{0 + \sqrt{x + \ldots}}}}}$. If $f(a) = 4$, what is $f'(a)$?

A) $\frac{4}{31}$ B) $\frac{31}{4}$ C) 1 D) 0 E) $\frac{1}{4}$

39. A person starting with \$64 and making 6 bets, wins 3 times and loses 3 times, the wins and losses occurring in random order. The chance of a win is equal to the chance of a loss. If each wager is for half the money remaining at the time of the bet, then the final result is

A) A loss of \$27
B) A gain of \$27
C) A loss of \$37
D) Neither a gain or a loss
E) A gain or a loss depending on the order in which the wins and losses occur

40. Let $\Diamond$ be a binary operation on the set of the positive real numbers that satisfies the following rules:

$$(x \cdot y^2) \Diamond y = x(y \Diamond 1) \text{ and } (x \Diamond 1) \Diamond x = 1$$

If $1 \Diamond 1 = 1$, what is $x \Diamond y$?

A) xy B) $\frac{y}{x}$ C) $\frac{x}{y}$ D) $x^2 y$ E) xy^2