TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

FIFTY-NINTH ANNUAL MATHEMATICS CONTEST

2015

Calculus and Advanced Topics

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Scoring formula: 4 x (Number Right) – (Number Wrong) + 40

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you; if you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

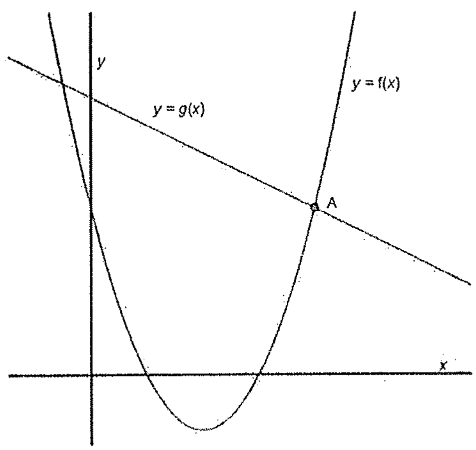
When told to do so, open your test booklet and begin. You will have exactly eighty minutes to work.

1. For what value of the constant c is the function $f(x)$ continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 3 \\ x^2 - cx & \text{if } x \geq 3 \end{cases}$$

a. There is no such value c.
b. 1
c. 0
d. $\frac{2}{3}$
e. $\frac{1}{4}$

2. The graphs below represent two functions, $y = f(x)$ and $y = g(x)$. Point A has coordinates (8, 6) and lies on the intersection of the two functions. Consider the following statements about the given functions.



Which of the following statements is true?

a. $f(6) = g(6)$
b. $f(8) = g(8)$
c. $f(6) = g(8)$
d. $f(8) = g(6)$
e. For any value of x, it is impossible to know the relationship between $f(x)$ and $g(x)$ without knowing the equations of the functions.

3. The population P of a certain colony of bacteria t days after the initial observation is given by $P = 1100(6)^{t/7}$. The time in days required for the population to triple is

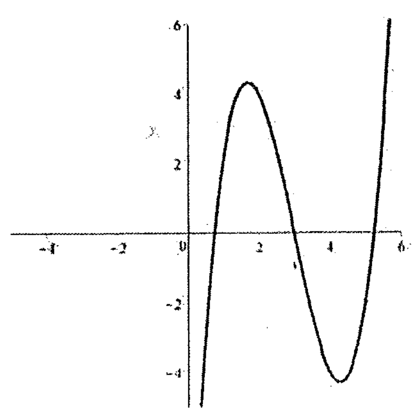
a. $7\log_6 3$
b. $2\log_7 6$
c. $6\log_6 3$
d. $7\log_2 6$
e. $6\log_7 3$

4. Calculate the limit.

$$\lim_{x\to5}\frac{x^2-3x-10}{x^2+2x-35}$$

a. $\frac{7}{12}$
b. $\frac{2}{7}$
c. 0.583
d. 5
e. 1

5. The graph of a function $y = f(x)$ is shown. Approximate the value of $f'(3)$.



a. 0
b. -1
c. 1
d. -5
e. 5

6. How many critical points does the function $f(x) = |x^2 - 1|$ have?

a. 0
b. 1
c. 2
d. 3
e. 4

7. If f is a continuous function and $F'(x) = f(x)$ for all real numbers x, then $\int_3^9 f(\frac{x}{3})dx =$

a. $\frac{1}{3}F(9) - \frac{1}{3}F(3)$
b. $3F(9) - 3F(3)$
c. $3F(3) - 3F(1)$
d. $\frac{1}{3}F(3) - \frac{1}{3}F(1)$
e. $\frac{1}{3}F(27) - 3F(9)$

8. The area bounded by the curves $f(x) = x$ and $f(x) = x^3$ is

a. 1
b. $\frac{1}{2}$
c. 0
d. $\frac{1}{4}$
e. $\frac{3}{4}$

9. Find all discontinuities of the function $f(x) = \frac{x^2-3x}{120x^2+x}$ and classify each discontinuity as removable or non-removable.

a. Removable: $x = 0$; Non-removable: $x = 3,\ x = -\frac{1}{120}$
b. Removable: $x = -\frac{1}{120}$; Non-removable: $x = 0$
c. Removable: $x = 0$; Non-removable: $x = -\frac{1}{120}$
d. Removable: $x = 3,\ x = -\frac{1}{120}$; Non-removable: $x = 0$
e. Removable: $x = 3$; Non-removable: $x = -\frac{1}{120}$

10. Consider the function $f(x) = x^3 - 5$ on the interval $[0, 4]$. Find the value(s) guaranteed by the Mean Value Theorem.

a. $x = \pm 2$
b. $x = 2$
c. $x = \pm\frac{4\sqrt{3}}{3}$
d. $x = \frac{4\sqrt{3}}{3}$
e. The Mean Value Theorem does not apply to this function on the given interval.

11. Differentiate $f(x) = \int_0^{x^2} 3^{\sin t}\,dt$.

a. $f'(x) = 3^{\sin t}$
b. $f'(x) = 3^{\sin x}$
c. $f'(x) = 3^{\sin x^2}$
d. $f'(x) = (2x)3^{\sin x^2}$
e. $f'(x) = (2x)3^{\sin x^2} \ln 3$

12. Let f and g be functions which are differentiable on the open interval $(0, 6)$, and let the values of f and g as well as their respective derivatives be given by the table below:

	$x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 5$
$f(x)$	3	4	1	5	2
$f'(x)$	4	3	2	5	1
$g(x)$	3	4	5	2	1
$g'(x)$	5	3	4	1	2

Find $C'(2)$, where $C(x) = (f \circ g)(x)$.

a. 15
b. 12
c. 9
d. 6
e. 5

13. What is the acute angle of separation (in degrees) between the lines $y = x$ and $y = 2x$? Round your answer to the nearest hundredth of a degree.

a. 18.27°
b. 18.43°
c. 18.68°
d. 18.92°
e. 19.23°

14. Which of the following is equal to $\int_a^b f(t)\,dt$ for any continuous function f?

a. $\int_{2a}^{2b} f(2u)\,du$
b. $\int_{2a}^{2b} \frac{1}{2} f(2u)\,du$
c. $\int_{a/2}^{b/2} f(2u)\,du$
d. $\int_{2a}^{2b} 2f(2u)\,du$
e. $\int_{a/2}^{b/2} 2f(2u)\,du$

15. Find all horizontal asymptotes of the function $p(x) = \frac{\sqrt{2x^2-5}}{943x+178}$.

a. $y = -\frac{178}{943}$
b. $y = \frac{\sqrt{2}}{943}$
c. $y = \frac{2}{943}$
d. $y = \pm\frac{\sqrt{2}}{943}$
e. $y = \pm\frac{2}{889249}$

16. Calculate the limit:

$$\lim_{x\to 0}\left(\cos x - \frac{\sin(2x)}{x}\right) e^{\left(\cos x - \frac{\sin(2x)}{x}\right)}$$

a. $\frac{1}{2}e^{1/2}$
b. $-e$
c. 0
d. $-\frac{1}{e}$
e. The limit does not exist.

17. For which values of x is $f(x) = \ln(2x^2 + 1)$ concave up?

a. $(-\frac{1}{2}, \frac{1}{2})$
b. $\left(-\infty, -\frac{1}{2}\right), \left(\frac{1}{2}, \infty\right)$
c. $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
d. $\left(-\infty, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \infty\right)$
e. The function is never concave up.

18. The length of one of the legs of a right triangle is increasing at a rate of 3 m/sec, while the length of the other leg is decreasing at a rate of 2 m/sec. At what rate is the hypotenuse changing when both legs have a length of 20 m?

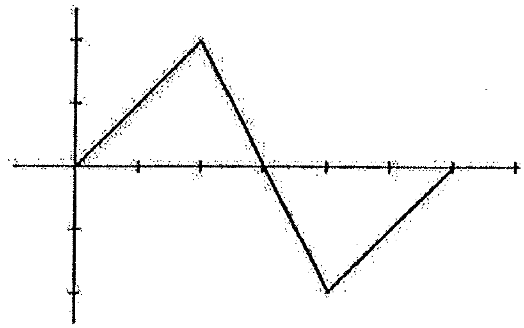
a. $\sqrt{13}$ m/sec
b. $\frac{\sqrt{13}}{2}$ m/sec
c. $\frac{\sqrt{2}}{2}$ m/sec
d. $\frac{5\sqrt{2}}{2}$ m/sec
e. $20\sqrt{2}$ m/sec

19. Compute the limit.

$$\lim_{n\to\infty} \frac{(2n+2)!n!}{(n+2)!(2n)!}$$

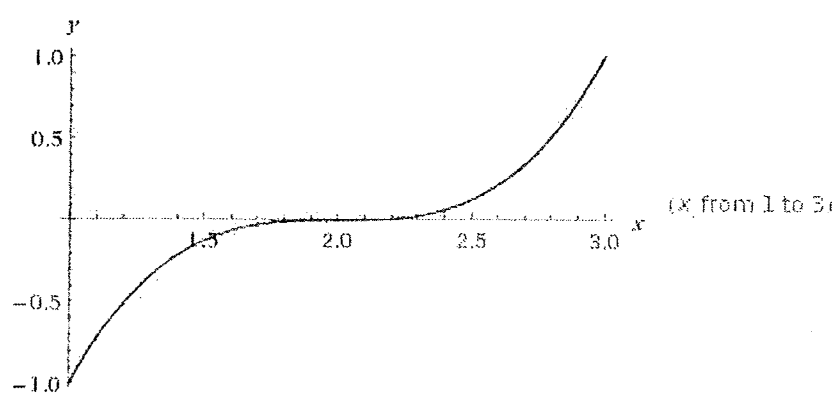
a. 0
b. 1
c. 2
d. 4
e. ∞

20. The graph of f' is given below. On what interval(s) is $f(x)$ concave down? Each axis is marked off in intervals of length 1.



a. $(0,2) \cup (4,6)$
b. $(2,4)$
c. $(0,3)$
d. $(3,6)$
e. The graph of $f(x)$ is never concave down.

21. The graph of f' is given below. This graph has a horizontal tangent line at $x = 2$. What feature must the function f have at $x = 2$?



a. an inflection point
b. a local maximum that the Second Derivative Test cannot detect
c. a local maximum that the Second Derivative Test can detect
d. a local minimum that the Second Derivative Test cannot detect
e. a local minimum that the Second Derivative Test can detect

22. A cylindrical tank is used as a batch chemical reactor. If the reaction being carried out is a simple isomerization reaction ($A \rightarrow R$), the concentration of product R at time t may be written as

$$R(t) = C(1 - e^{-kt})$$

where $R(t)$ is the concentration of product R at time t, k is a positive constant known as the reaction rate constant, and C is the initial concentration of reactant A. Suppose $C = 5$ moles per liter and $k = 0.1$ /sec. What is the expected concentration of product as $t \rightarrow \infty$?

a. 0 moles per liter
b. 0.1 moles per liter
c. 5 moles per liter
d. 0.5 moles per liter
e. 10 moles per liter

23. Find the sum.

$$\sum_{k=1}^{10000} \binom{10000}{k}.$$

a. 2^{10000}
b. 2^{10001}
c. $2^{10000} + 1$
d. $2^{10000} - 1$
e. $2^{10001} - 1$

24. Let f and g be functions which are positive and differentiable for all real numbers x, with the additional property that $f(0) = 1$.
If $h(x) = f(x)g(x)$ and $h'(x) = h(x) + f(x)g'(x)$, then $f(x)$ must equal

a. $g(x)$
b. $g'(x)$
c. $h(x)$
d. $1 + \ln(x + 1)$
e. e^x

25. Suppose the infinite sequence $a_1, a_2, a_3, \ldots$ converges to 30000. Create a new sequence by multiplying the first 1000 terms of this sequence by 2, resulting in the sequence $b_1, b_2, b_3, \ldots$, where $b_n = a_n$ when $n > 1000$ and $b_n = 2a_n$ when $n \leq 1000$. Then the sequence $b_1, b_2, b_3, \ldots$

a. converges to 30000.
b. converges to 60000.
c. converges to some number in the open interval $(30000, 60000)$.
d. must diverge.
e. could either converge or diverge; there is not enough information.

26. Let $U = \{f(x) | f(x) \text{ is a polynomial}\}$. Define a binary operation $f * g$ on the set U by $f * g = \frac{d(fg)}{dx}$ for any two elements f, g in U. Which of the following is true?

a. The binary operation $*$ is both associative and commutative.
b. The binary operation $*$ is neither associative nor commutative.
c. The binary operation $*$ is associative but is not commutative.
d. The binary operation $*$ is commutative but not associative.
e. Not enough information is provided to make a decision about the binary operation.

27. Assume the earth is a sphere whose great circle has a circumference of 25,000 miles. The latitude of the city of Adamsville is $41° 8'$. The city of Bakersfield is 500 miles due south of Adamsville. What is the latitude of Bakersfield to the nearest minute?

a. $33° 48'$
b. $37° 56'$
c. $34° 6'$
d. $33° 56'$
e. $37° 48'$

28. Which of the following is the best approximation for

$$\sum_{x=1}^{5000}(2x^2 - 5x + 3)?$$

a. 1.15×10^{10}
b. 3.13×10^{10}
c. 6.25×10^{10}
d. 8.33×10^{10}
e. 9.27×10^{10}

29. If $\frac{\pi}{4} < a < \frac{\pi}{2}$, find $\int_{\frac{\pi}{4}}^{a} \sec x \; dx$.

a. $\ln|\cos a + \tan a| - \ln\left(1 + \frac{\sqrt{2}}{2}\right)$
b. $\ln|\cos a| - \frac{\sqrt{2}}{2}$
c. $\csc a - \sqrt{2}$
d. $\ln|\sin a| - \ln\left(\frac{\sqrt{2}}{2}\right)$
e. $\ln|\sec a + \tan a| - \ln(1 + \sqrt{2})$

30. Suppose each of the functions f and g is positive and differentiable for every real number x. Then the derivative of $f(x)^{g(x)}$ is

a. $g(x)f(x)^{g(x)-1}$
b. $g(x)f(x)^{g(x)-1}f'(x)$
c. $g(x)f(x)^{g(x)-1}f'(x) + f(x)^{g(x)}g'(x)$
d. $f(x)^{g(x)}g'(x)\ln f(x)$
e. $g(x)f(x)^{g(x)-1}f'(x) + f(x)^{g(x)}g'(x)\ln f(x)$

31. Which point on the unit circle and in the fourth quadrant has a tangent line which contains the point $(4, 4)$? Round your coordinates to the nearest thousandth.

a. $(0.715, -0.699)$
b. $(0.684, -0.729)$
c. $(0.945, -0.326)$
d. $(0.660, -0.751)$
e. $(0.821, -0.571)$

32. Consider the torus ("doughnut" shape) generated by letting the unit circle revolve about the line $x = 5$. The volume of this torus is

a. 5π
b. 10π
c. $5\pi^2$
d. $10\pi^2$
e. $25\pi^2$

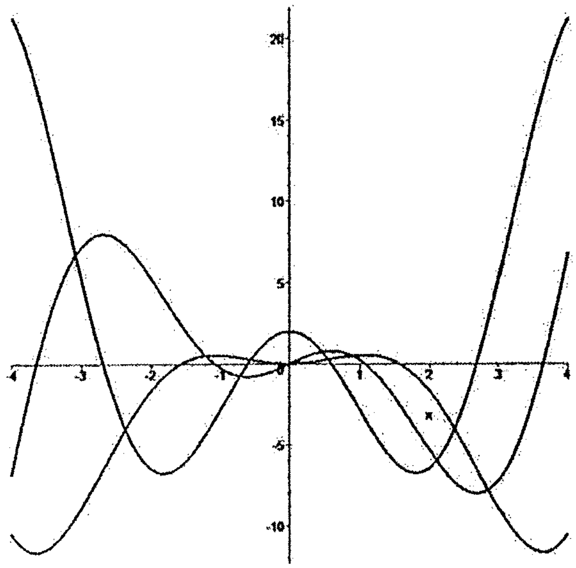
33. The energy loss due to friction in a certain pipe may be written as

$$h(v) = 0.7459v^2 f(v)$$

where v is the average velocity in the pipe and $f(v)$ is called the Fanning Friction Factor. Suppose an engineer knows through experimentation that $f(6) = 0.008$, $f(7) = 0.009$, and $f'(x) > 0$ on $(6, 7)$. Approximate $h(6.3)$. Round your approximation to the nearest ten-thousandth.

a. 0.2457
b. 0.0158
c. 0.2364
d. 0.0083
e. 0.2282

34. The graphs of f, f', and f'' on [−4, 4] are shown below. How many horizontal tangents does f have for x in [−4, 4]?



a. 1
b. 2
c. 3
d. 4
e. 5

35. Some values of a cubic function f are given in the following table. Determine $\int_{-3}^{3} f(x)dx$.

x	$f(x)$
-3	5.8
-2	9.2
0	10
1	8.6
3	5.2

a. 44.0
b. 48.2
c. 51.0
d. 51.4
e. 56.4

36. Let $f(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$. Find $f'(x)$ at the positive value $x = c$ satisfying $f(c) = 2$.

a. $\frac{\sqrt{2}}{1+\ln\sqrt{2}}$

b. $\frac{2\sqrt{2}}{1+2\ln\sqrt{2}}$

c. $\frac{2}{1-2\ln\sqrt{2}}$

d. $\frac{2\sqrt{2}}{1-2\ln\sqrt{2}}$

e. $\frac{\sqrt{2}}{1-2\ln\sqrt{2}}$

37. Use the tangent line to the graph of $f(x) = \sqrt[5]{x}$ at the point $x = 32$ to approximate $\sqrt[5]{40.9}$. Round your answer to the nearest ten-thousandth.

a. 2.0913
b. 2.1006
c. 2.1016
d. 2.1113
e. 2.1118

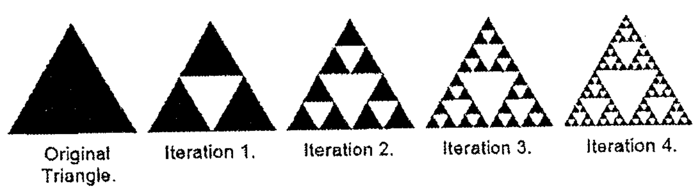
38. The Sierpinski triangle is a figure constructed from an equilateral triangle by repeatedly removing triangular subsets. To construct the Sierpinski triangle:

Step 1: Begin with an equilateral triangle.

Step 2: Subdivide the triangle into four smaller congruent equilateral triangles and remove the central one.

Step 3: Repeat Step 2 with each of the remaining smaller triangles.

The first few iterations of this process are illustrated below.



Let A represent the area of the original triangle (shaded in black). What is the area of the figure (shaded in black) in Iteration 25 in terms of A? Round your answer to the nearest ten-thousandth.

a. 0.0005 A
b. 0.0006 A
c. 0.0007 A
d. 0.0008 A
e. 0.0009 A

39. Let a and b be positive real numbers. Evaluate:

$$\lim_{x \to 0} (1 + ax)^{b/x}.$$

a. a^b
b. e^{ab}
c. $e^{a/b}$
d. b^a
e. $e^{b/a}$

40. Let $f(x) = \frac{e^{2x} - e^{-2x}}{2}$. What is the thirty-fifth derivative of $f(x)$?

a. $2^{34}(e^{2x} - e^{-2x})$
b. $2^{34}(e^{2x} + e^{-2x})$
c. $2^{35}(e^{2x} + e^{-2x})$
d. $2^{35}(e^{2x} - e^{-2x})$
e. $2^{36}(e^{2x} + e^{-2x})$