

TWENTY-FIRST ANNUAL MATHEMATICS CONTEST
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ADVANCED TOPICS TEST

1977

Scoring Formula: $4R - W$

EDITED BY:

Billy Edwards
and
John W. Jayne
The University of Tennessee
at Chattanooga,
Chattanooga, Tennessee

This test was prepared from a list of Advanced Topics questions submitted by Middle Tennessee State University.

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem there are listed 5 possible answers; one and only one is correct. You are to work each problem, determine the correct answer, and indicate your choice by making a heavy black mark in the correct place on the separate answer sheet provided. You must use a pencil with soft lead (No. 2 lead or softer). A sample problem follows:

1. If $2x = 3$, then x equals
- | | A | B | C | D | E |
|--------------|--------------------|---------|---|---|---|
| (a). $2/3$. | (b). 3. | (c). 6. | | | |
| (d). $3/2$. | (e). none of these | | | | |
1.

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The correct answer for the sample problem is $3/2$, which is answer (d); so you would answer this problem by making a heavy black mark under space D as indicated above.

This test has been constructed so that most of you are not expected to answer all questions. Do your very best on the questions you feel you know how to work. You will be penalized for incorrect answers so it is advisable not to do much wild guessing.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet.

The answer sheets will be used for a statistical compilation and will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will be able to keep this booklet after the test is completed.

When told to do so, open your test booklet to page 1 and begin. When you have finished one page, go on to the next. The working time for the entire test is 80 minutes.

1. Let f be a real polynomial and let $g(x) = \left[\frac{f(x)-f(0)}{x} \right]$, then $\lim_{x \rightarrow 0} g(x)$ is

- (a). $f(0)$
- (b). 0
- (c). $\frac{f(x)}{x}$
- (d). $f'(0)$
- (e). undefined

2. Assume that each person in Aplington has three initials, one for each of his first, middle, and last name. How many residents must Aplington have before we can be sure 2 residents have the same initials?

- (a). 32
- (b). $26^3 + 1$
- (c). $26^3 + 26^3$
- (d). 26^3
- (e). none of these

3. Find the domain and range of this function: $y = \sqrt{4-x^2}$.

- (a). D: $x \in \text{Reals}$; R: $y \geq 0$
- (b). D: $-2 \leq x \leq 2$; R: $0 \leq y \leq 2$
- (c). D: $-2 \leq x \leq 2$; R: $-2 \leq y \leq 2$
- (d). D: $|x| \geq 2$; R: $|y| \geq 2$
- (e). none of these

4. The binomial coefficient $\binom{n}{k}$ is defined by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
 What is the value of $\binom{n}{k-1} + \binom{n}{k}$?

(a). $\binom{2n}{2k-1}$

(b). $\binom{n}{k}$

(c). $\binom{n+1}{k}$

(d). $\binom{2n}{k}$

(e). none of these

5. Let $K = \{(x,y) : Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\}$. Which of the following is false?

(a). $B^2 - 4AC$ is invariant under rotation about 0.

(b). $B^2 - 4AC$ is invariant under a translation.

(c). If $B^2 - 4AC = 0$, then K is a parabola, two parallel lines, a line or ϕ .

(d). If $B^2 - 4AC < 0$, then the locus is a circle, an ellipse, a point or ϕ .

(e). none of these

6. If $0 \leq t < \frac{\pi}{2}$, then $\frac{1 - \sin t}{\cos t}$ is

(a). $\frac{\cot t}{1 + \csc t}$

(b). $\frac{\tan t}{1 + \csc t}$

(c). $\frac{\cot t}{\csc t - 1}$

(d). $\frac{\cos t}{1 - \sin t}$

(e). none of these

7. The angle between the hands of a clock at 1:20 p.m. is
- (a). 90°
 - (b). 100°
 - (c). 80°
 - (d). 60°
 - (e). none of these
8. The solution set for $S = \{x \mid x^2 - x - 2 < 0\}$ is
- (a). $\{x \mid x > 2\}$
 - (b). ϕ
 - (c). $\{x \mid x < -1 \text{ or } x > 2\}$
 - (d). $\{x \mid -1 < x < 2\}$
 - (e). all real numbers
9. The slope of the curve $y = x^3 - 6x + 2$ where it crosses the y-axis is
- (a). 0
 - (b). 4
 - (c). 6
 - (d). -6
 - (e). undefined
10. Given the points $P_1(3, 2)$ and $P_2(-3, -3)$. The coordinates of the midpoint of the line segment P_1P_2 are
- (a). $(0, -\frac{1}{2})$
 - (b). $(3, \frac{5}{2})$
 - (c). $(0, \frac{1}{2})$
 - (d). $(-\frac{1}{2}, -\frac{2}{3})$
 - (e). none of these

11. If a and b are real, then $|a| < |b|$ if and only if
- (a). $a < b$
 - (b). $a^2 < b^2$
 - (c). $\frac{1}{a} < \frac{1}{b}$
 - (d). $|a| < b$
 - (e). $\frac{1}{a} > \frac{1}{b}$
12. $\lim_{\theta \rightarrow 0} \left[\left(\frac{\sec \theta + 1}{\theta} \right) \left(\frac{\sec \theta - 1}{\theta} \right) \right]$ is
- (a). undefined
 - (b). 0
 - (c). 1
 - (d). 4
 - (e). infinite
13. Which of the following does not hold for all values of x where the functions are defined?
- (a). $0 \leq \sin^2 x \leq 1$
 - (b). $\sec^2 x \geq 1$
 - (c). $|\sin x| \leq |\tan x|$
 - (d). $x + \cos 2x$ is a monotone, nondecreasing function.
 - (e). none of the above
14. Find the distance from the point $(-2, 2)$ to the line $4x - 3y + 5 = 0$
- (a). $9/5$
 - (b). $\sqrt{3}$
 - (c). $7/4$
 - (d). $\frac{\sqrt{13}}{2}$
 - (e). $3/2$

15. The inner product of the vectors $(3, -2)$ and $(2, 5)$ is
- $(6, -10)$
 - $(-4, 11)$
 - 7
 - 4
 - none of the above
16. If $0 \leq A < \frac{\pi}{2}$, then $\frac{\cos 2A}{1+\sin 2A}$ is
- $\sin^2 A - \cos^2 A$
 - $2 \sin A \cos A + 1$
 - $\frac{1-\tan A}{1+\tan A}$
 - $\sin^3 A$
 - none of these
17. A box contains tickets numbered 1, 2, 3, ..., 10. If 2 tickets are drawn what is the probability that the sum of their numbers is even?
- 1/2
 - 5/9
 - 4/9
 - 8/9
 - 4/18
18. Solve: $1-2 \sin^2 x - \cos x = 0; 0 < x < 2\pi$
- $x = 0, \frac{5\pi}{6}, \frac{4\pi}{3}$
 - $x = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}$
 - $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$
 - $x = 0, -\frac{2\pi}{3}, \frac{4\pi}{3}$
 - none of these

19. $2\sqrt{2} \cdot \sqrt[4]{2} \cdot \sqrt[8]{4}$ is
- (a). $2\sqrt[13]{2}$
 - (b). $2\sqrt[14]{16}$
 - (c). 8
 - (d). 4
 - (e). 256
20. If the degrees of the angles of a triangle are in the ratio 2:3:7, the triangle is
- (a). acute
 - (b). isosceles
 - (c). obtuse
 - (d). right
 - (e). equilateral
21. What are all the values of y for which $y^2 < 9$?
- (a). $y < 3$
 - (b). $y > -3$
 - (c). $y < 9$
 - (d). $y > 9$
 - (e). $-3 < y < 3$
22. Solve: $(x+2)(x-3) = 10$
- (a). 8, 13
 - (b). 8, 7
 - (c). $\frac{1}{2} \pm \sqrt{65}$
 - (d). $\frac{1}{2} \pm \frac{1}{2}\sqrt{65}$
 - (e). cannot be solved

23. Which of the following is one solution to $4 \sin 2x \cos 2x = \sqrt{3}$.
- (a). 240°
 - (b). 15°
 - (c). 60°
 - (d). $\frac{\sqrt{3}}{2}$
 - (e). $\frac{\sqrt{3}}{4}$
24. The surface area of a cube is 150 square feet. How many cubic feet are in the volume of the cube?
- (a). 30
 - (b). 50
 - (c). 100
 - (d). 125
 - (e). 150
25. What real number most exceeds its square?
- (a). $1/4$
 - (b). $1/8$
 - (c). $.40$
 - (d). $.75$
 - (e). none of these
26. From 9:00 a.m. to 2:00 p.m. the temperature rose at a constant rate from -14°F to 36°F . What was the temperature at noon?
- (a). -4°
 - (b). $+6^\circ$
 - (c). $+16^\circ$
 - (d). $+26^\circ$
 - (e). $+31^\circ$

27. Four people are chosen at random. The probability of at least 2 of them having birthdays on the same day of the week (Monday, Tuesday, etc.) is

- (a). 12/35
- (b). 20/49
- (c). 30/49
- (d). 120/343
- (e). 223/343

28. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$ is

- (a). undefined
- (b). 6
- (c). 0
- (d). 3
- (e). $x \neq 3$

29. $\frac{x}{x^{1/3} - 1} - \frac{x^{2/3}}{x^{1/3} + 1} - \frac{1}{x^{1/3} - 1} + \frac{1}{x^{1/3} + 1}$ is

- (a). $x^{2/3} + 2$
- (b). $x^{4/3} + 2$
- (c). $x^{4/3} - 1$
- (d). $x^{2/3}$
- (e). $x^{2/3} - 2$

30. The value printed by the program

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10 Let A = 1
20 Let B = 1
30 For J = 1 to 7
40 Let B = B + A
50 Let A = A + B
60 Next J
70 Print A
80 End

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- (a). 377
- (b). 2
- (c). 1
- (d). 610
- (e). 987

31. A bug leaves the origin and crawls 1 unit north, $\frac{1}{2}$ unit east, $\frac{1}{4}$ unit south, $\frac{1}{8}$ unit west, $\frac{1}{16}$ unit north, and so on. Find the coordinates of his ultimate destination.
- (a). $(4/5, 2/5)$
 - (b). $(1/5, 4/5)$
 - (c). $(2/5, 4/5)$
 - (d). $(2/5, 3/5)$
 - (e). $(3/7, 4/7)$
32. The number of committees of three persons which can be formed from a group of eight individuals is
- (a). 10
 - (b). 40
 - (c). 24
 - (d). 56
 - (e). none of these
33. The Newton-Raphson Method may fail in approximating a root of an equation if the root sought is an inflection point of the curve because
- (a). the root may be imaginary
 - (b). you cannot draw a tangent to this particular point
 - (c). the successive tangents may "box" the root
 - (d). the curve does not cross at an integral value
 - (e). none of these
34. The solution of the nonlinear system $\begin{cases} x^2 + 2y^2 = 1 \\ x + y = 2 \end{cases}$ is
- (a). $(1, 1)$
 - (b). $(3, -1)$
 - (c). $(0, 0)$
 - (d). $(4, 2)$
 - (e). none of these

35. The equation $x^2 - y^2 - 4x + 6y - 21 = 0$ represents:
- (a). a circle with center $(2, 3)$
 - (b). an ellipse with center $(2, 3)$
 - (c). a hyperbola with vertices at $(-2, 3), (6, 3)$
 - (d). a hyperbola with center $(-2, 3)$
 - (e). none of these
36. The area of a right triangle is 24 square inches. The ratio of its legs is 2:3. Find the number of inches in the hypotenuse of the right triangle.
- (a). $2\sqrt{13}$
 - (b). $\sqrt{26}$
 - (c). $4\sqrt{13}$
 - (d). $\sqrt{13}$
 - (e). $\sqrt{104}$
37. $\tan(\text{Arcsin}(-\frac{1}{2}))$ is
- (a). $-\frac{1}{\sqrt{3}}$
 - (b). $-\frac{\pi}{6}$
 - (c). $-\sqrt{3}$
 - (d). $\sqrt{3}/3$
 - (e). $\frac{7\pi}{6}$
38. If x is any real number, then for all x , $\sin x \cos x = 0$ is equivalent to
- (a). $\sin x = 0$
 - (b). $\sin 2x = 0$
 - (c). $\sqrt{1-\sin^2 x} = 0$
 - (d). $\cos x = 0$
 - (e). none

39. The following is sometimes given as a "proof" that $1 = -1$.

"Proof"

$$1 = \sqrt{1}$$
$$\sqrt{1} = \sqrt{(-1)(-1)}$$
$$\sqrt{(-1)(-1)} = \sqrt{(-1)^2} = -1$$
$$\therefore 1 = -1$$

Which of the statements in the "proof" is invalid?

- (a). $\sqrt{1} = 1$
- (b). $\sqrt{1} = \sqrt{(-1)(-1)}$
- (c). $\sqrt{(-1)(-1)} = -1$
- (d). all of the above
- (e). none of the above
40. Evaluate the limit: $\lim_{x \rightarrow 5} \frac{x^3 - 4x^2 - 7x + 10}{x - 5}$

- (a). no limit exists
- (b). 0
- (c). 11
- (d). 28
- (e). 70