

THIRTY-FIFTH ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Advanced Topics II 1991

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Scoring formula: $4R - W + 40$

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DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer, and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all the questions. Do your very best on the questions you feel you know how to work. You will be penalized for incorrect answers, so it is advisable not to do wild guessing.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet. The answer sheets will not be returned to you. If you wish to have a record of your performance, mark your answers in this booklet also. You will be able to keep this booklet after the test is completed.

When told to do so, open your test booklet and begin. The working time for the entire test is 80 minutes.

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1. $\frac{1-i}{1+i} = ?$
- a) 2 b) $-i$ c) -2 d) i e) 0
2. The natural domain of the function f defined by $f(x) = \sqrt{9-x^2}$ is
- a) $[3, \infty)$ b) $(-3, 3)$ c) $[-3, 3]$ d) $(-\infty, 3]$ e) $[-3, \infty)$
3. If $f(x) = \sqrt{x^2-1}$ and $g(x) = 1-x$, then $|(f \circ g)(2)| = ?$
- a) 0 b) $1 - \sqrt{3}$ c) $\sqrt{3} - 1$ d) 2 e) 4
4. $\frac{2}{3} + \frac{1}{8}\left(\frac{9}{10}\right) + \frac{1}{8}\left(\frac{9}{10}\right)^2 + \dots + \frac{1}{8}\left(\frac{9}{10}\right)^n + \dots = ?$
- a) $\frac{19}{24}$ b) $\frac{9}{8}$ c) $\frac{23}{12}$ d) $\frac{187}{240}$ e) $\frac{43}{24}$

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5. The slope of the tangent line to the graph of $y = \tan^{-1}(x/4)$ at the point $(4, \frac{\pi}{4})$ is

- a) $\frac{1}{68}$ b) $\frac{1}{17}$ c) $\frac{1}{2}$ d) $\frac{1}{8}$ e) $\frac{1}{4\cos^2(1)}$

6. The value of $\int_{-1}^{1/3} \frac{dx}{9x^2 + 6x + 5}$ is

- a) $\frac{\pi}{10}$ b) $\frac{\pi}{7}$ c) $\frac{\pi}{6}$ d) $\frac{\pi}{12}$ e) $\frac{\pi}{9}$

7. If $x = 3\sin 2t$ and $y = 2\cos 3t$, then $\frac{dy}{dx} = ?$

- a) $\frac{-3}{2}\tan t$ b) $\frac{-\sin 3t}{\cos 2t}$ c) $\frac{2}{3}\cot t$ d) $\frac{-\cos 2t}{\sin 3t}$ e) $\frac{\sin 3t}{\cos 2t}$

8. The region bounded by $y = x$, $y = 2$ and $x + y = 2$ is revolved about the x-axis. The volume of the resulting solid revolution is

- a) $\frac{10\pi}{3}$ b) $\frac{5\pi}{3}$ c) $\frac{16\pi}{3}$ d) 2π e) $\frac{32\pi}{3}$

ADVANCED TOPICS II

9. The position of a particle at time t is given by $s(t) = |-3t + 2|$. The velocity of this particle at $t = \frac{2}{3}$ is

- a) 3 b) -3 c) 0 d) undefined e) 5

10. The value of $\sin\left(\arctan\frac{1}{4}\right)$ is

- a) $\frac{1}{4}$ b) $\frac{1}{\sqrt{17}}$ c) $\frac{4}{\sqrt{15}}$ d) $\frac{4}{\sqrt{17}}$ e) $\frac{1}{\sqrt{15}}$

11. The inverse of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is given by

- a) $\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ d) $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ e) the inverse doesn't exist

12. Which of the following vectors is perpendicular to $\langle 1, 2 \rangle$?

- a) $\langle -1, -2 \rangle$ b) $\langle 2, 1 \rangle$ c) $\langle 4, -2 \rangle$ d) $\langle -2, -1 \rangle$ e) $\langle 4, 2 \rangle$

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13. The smallest positive integer n such that $A^n = I$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is

- a) $n = 5$ b) $n = 2$ c) $n = 3$ d) $n = 4$ e) $n = 7$

14. A parallelogram has vertices $(0,0)$, $(1,\sqrt{3})$, $(4,0)$. A fourth vertex and the area are given by

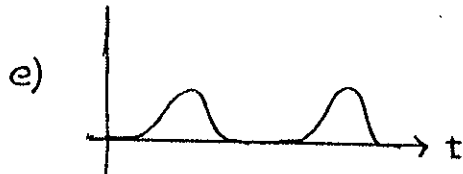
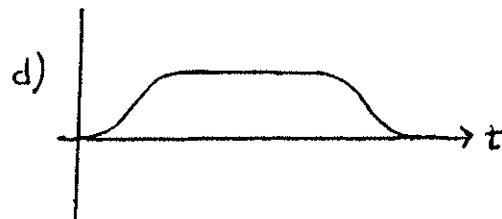
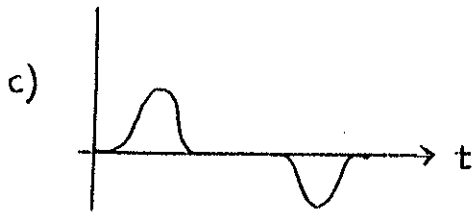
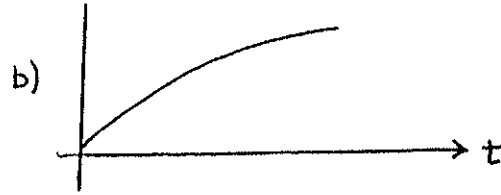
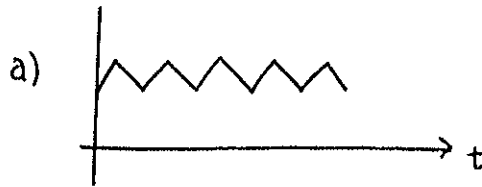
- a) $(5,\sqrt{3})$ & $4\sqrt{3}$ b) $(4,\sqrt{3})$ & 10
 c) $(5,\sqrt{3})$ & $5\sqrt{3}$ d) $(4,1)$ & $5\sqrt{3}$
 e) $(16,3)$ & 8

15. What is $\frac{d}{dx} \int_{x^2}^{x^3} e^{t^2/2} dt$

- a) $\int_{x^2}^{x^3} te^{t^2/2} dt$ b) $\int_{2x}^{3x^2} e^{t^2/2} dt$ c) $e^{x^6/2} - e^{x^4/2}$
 d) $3x^2 e^{x^6/2} - 2xe^{x^4/2}$ e) 0

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16. The following graphs claim to depict acceleration as a function of time for an elevator traveling from the ground floor to the 20th floor of a high rise office building. Choose the graph closest to describing this trip.



17. What is $\lim_{x \rightarrow \infty} \sqrt{x^2 - 4x} - x$?

- a) 0 b) -4 c) $-\infty$ d) -2 e) 4

18. To what does the series $\sum_{n=0}^{\infty} nr^{n-1}$ converge ($0 < r < 1$)?

- a) $\frac{1}{1-r}$ b) $\frac{1}{(1-r)^2}$ c) ∞ d) $\frac{n}{1-r}$ e) $\frac{r}{1-r^2}$

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19. The tangent line to the curve $x^2 + xy + y^2 = 7$ at the point $(2,1)$ has slope
- a) 1 b) $\frac{-5}{2}$ c) $\frac{-5}{4}$ d) -3 e) -1
20. The number of real numbers x which satisfy the equation $\sin x = x^4 - 1$ is
- a) 0 b) 1 c) 2 d) 3 e) 4
21. Which of the following equations holds for all real θ
- a) $\sin 3\theta = 3\sin\theta\cos^2\theta + \sin^3\theta$ b) $\sin 3\theta = \sin\theta\cos^2\theta - \sin^3\theta$
c) $\sin 3\theta = \sin\theta\cos^2\theta + \sin^3\theta$ d) $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$
e) $\sin 3\theta = 3\sin\theta$
22. The volume of the solid obtained by revolving $e^{-x^2/2}$, $0 \leq x < \infty$, about the y -axis is given by
- a) $2\pi e$ b) 2π c) 0 d) ∞ e) $\frac{1}{2}\pi^{3/2}$

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23. The equation of the line tangent to the graph of the function $f(x) = \int_1^x \frac{1}{t} dt$ passing through the point $(1,0)$ is given by

a) $y = \ln(1)(x-1)$

b) $y = x + 1$

c) $y = x - 1$

d) $y = \ln(0)(x-0)$

e) $y = \frac{1}{x^2}(x-1)$

24. An urn contains spheres and cubes, each colored either orange or white. If 50% of the objects in the urn are spheres, 35% are orange, and 70% are orange or spherical, what is the probability that an object chosen at random from this urn is an orange sphere?

a) 0.175

b) 0.15

c) 0.20

d) 0.35

e) 0.70

25. How many even 3-digit numbers with no digits repeated are there? (it is understood that the first digit must be non zero)

a) 405

b) 720

c) 360

d) 328

e) 450

26. If the letters a, A, b, B, c, and C are arranged at random in a row, what is the probability that the lower case letters appear in alphabetical order?

a) $\frac{1}{6}$

b) $\frac{1}{2}$

c) $\frac{1}{720}$

d) $\frac{1}{36}$

e) $\frac{1}{30}$

ADVANCED TOPICS II

27. $\lim_{x \rightarrow 0^+} x^{2x}$ is

- a) 0 b) 1 c) 2 d) ∞ e) e

28. The equation of the parabola with vertex at $(-1,2)$ and focus at $(1,2)$ is

- a) $y^2 - 4y - 8x - 4 = 0$ b) $y^2 - 4y - 4x = 0$
c) $y^2 - 4y - 2x + 2 = 0$ d) $x^2 - 4x - 4y + 2 = 0$
e) $x^2 - 4x - 8y - 4 = 0$

29. The radius of the circle (in polar coordinates) $r = 2\cos\theta - \sin\theta$ is

- a) $\sqrt{5}/2$ b) $\sqrt{3}$ c) $2/\sqrt{3}$ d) 2 e) $\sqrt{2}$

30. Descartes, Locke, Berkeley, Hume, and Kant line up at random for a photograph. What is the probability that Hume and Kant are not standing next to each other?

- a) $\frac{1}{5}$ b) $\frac{1}{120}$ c) $\frac{2}{5}$ d) $\frac{4}{5}$ e) $\frac{3}{5}$

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31. A pair of dice is thrown 3 times. What is the probability that on at least one of the throws the sum of the spots showing is 11 ?
- a) $1 - \left(\frac{1}{11}\right)^3$ b) $1 - \left(\frac{10}{11}\right)^3$ c) $1 - \left(\frac{17}{18}\right)^3$ d) $\left(\frac{1}{18}\right)^3$ e) $\left(\frac{1}{11}\right)^3$
32. Consider the functions I. $f(x) = e^{\frac{1}{2}x + 3}$ II. $f(x) = \log x$ III. $f(x) = 10x + 5$
 IV. $f(x) = 17^x$ V. $f(x) = x^2$. For which of those functions is it true that the rate of change of the function is directly proportional to the value of the function at each point in the domain of the function?
- a) III and V b) II and III c) III
 d) I and IV e) I, III, and IV.
33. Consider the function $f: [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}] \rightarrow [-1, 1]$ defined by $f(x) = \sin x$. The function f is
- a) both injective and surjective b) injective only
 c) surjective only d) neither injective nor surjective
 e) not well-defined

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34. If $\lim_{n \rightarrow \infty} a_n = l$, which of the following statements is necessarily true?
- a) For all $\epsilon > 0$ there exists a positive integer N such that, if n and m are integers such that $n > N$ and $m > N$, then $|a_n - a_m| < \epsilon$.
 - b) There exists a positive integer N such that, for all $\epsilon > 0$, if n is an integer and $n > N$, then $|a_n - l| < \epsilon$.
 - c) There exists a positive integer N such that, if n is an integer and $n > N$, then $a_n = l$.
 - d) For all $\epsilon > 0$ and for every positive integer N , if n is an integer and $n > N$, then $|a_n - l| < \epsilon$.
 - e) The infinite series $\sum_n a_n$ converges.

ADVANCED TOPICS II

35. At least one of the following statements is true. Identify the strongest true statement among them.
- a) $\sqrt{2}$ is irrational.
 - b) \sqrt{p} is irrational for every prime number p .
 - c) $\sqrt[r]{p}$ is irrational for every prime number p and every positive integer $r \geq 2$.
 - d) $\sqrt[n]{r}$ is irrational for every positive integer $r \geq 2$ and every positive integer n that is not a perfect r th power.
 - e) If $p(x) = x^r + c_{r-1}x^{r-1} + \dots + c_1x + c_0$ where the coefficients c_0, c_1, \dots, c_{r-1} are all integers, then every root of $p(x)$ that is not an integer is irrational.
36. The value of $\int_0^{\pi} [1 + \cos x]^2 dx$ is
- a) $3\pi/2$
 - b) π
 - c) $\pi + 2$
 - d) $\pi - 2$
 - e) 2π
37. The set all points $P=(x,y)$ such that the distance from P to $(1,2)$ is twice the distance from P to $(5,3)$ is
- a) ellipse
 - b) parabola
 - c) hyperbola
 - d) circle
 - e) a single point

ADVANCED TOPICS II

38. Water is poured into a conical cup at the rate of 1 cubic inch per second. If the cup is 6 inches tall and the top of the cup has a diameter of 4 inches, how fast does the water level rise (inches per second) when the water level is 3 inches deep?
- a) 1 b) 3 c) $\frac{1}{\pi}$ d) $\frac{2}{\pi}$ e) $\frac{3}{\pi}$
39. If $f(x)=ax^3+bx^2+cx+d$, f has a local maximum of 4 at $x = -1$, and a local minimum of -2 at $x=1$, then the value of d is
- a) -2 b) -1 c) 0 d) 1 e) 2
40. Of all possible right circular cones which can be inscribed in a sphere of radius R , the volume of the one of maximum volume is
- a) $4\pi R^3/9$ b) $8\pi R^3/27$ c) $16\pi R^3/27$ d) $32\pi R^3/81$
e) $2\pi R^3/3$

