TENNESSEE MATHEMATICS TEACHERS ASSOCIATION

SIXTY-FIRST ANNUAL MATHEMATICS CONTEST

2017

Calculus and Advanced Topics

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Scoring formula: 4 x (Number Right) – (Number Wrong) + 40

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you; if you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly eighty minutes to work

1. Find the point at which the tangent line of the curve $f(x) = e^{5x} - 2x + 4$ is parallel to the line $3x - y = 1$.

   (a) $(0, f(0))$
   (b) $(1, f(1))$
   (c) $(2, f(2))$
   (d) $(-1, f(-1))$
   (e) does not exist

2. If the price of a product is given by $p(x) = \frac{2016}{x} + 2015$, where x represents the demand for the product, find the rate of change of price when the demand is 8.

   (a) 252
   (b) -252
   (c) 31.5
   (d) -31.5
   (e) 2016

3. Find the derivative of f.

$$f(x) = \ln(x^2 + 1 + e^x).$$

   (a) $f'(x) = (x^2 + 1 + e^x)/(2x + e^x)$
   (b) $f'(x) = (x^2 + e^x)/(x^2 + e^x + 1)$
   (c) $f'(x) = (2x + e^x)/(x^2 + e^x)$
   (d) $f'(x) = (2x - e^x)/(x^2 + e^x + 1)$
   (e) $f'(x) = (2x + e^x)/(x^2 + e^x + 1)$

4. Find the derivative of the function $f(t) = t^{\alpha-1}(1-t)^{\beta-1}$ for some constants $\alpha, \beta > 0$ on the interval $(0, 1)$.

   (a) $(\alpha - 1)t^{\alpha-2}(1-t)^{\beta-1} + (\beta - 1)t^{\alpha-1}(1-t)^{\beta-2}$
   (b) $(\alpha - 1)t^{\alpha-2}(1-t)^{\beta-1} - (\beta - 1)t^{\alpha-1}(1-t)^{\beta-2}$
   (c) $(\alpha - 1)t^{\alpha-1}(1-t)^{\beta-1} + (\beta - 1)t^{\alpha-1}(1-t)^{\beta-1}$
   (d) $(\alpha - 1)t^{\alpha-1}(1-t)^{\beta-1} - (\beta - 1)t^{\alpha-1}(1-t)^{\beta-1}$
   (e) $(\alpha - 1)t^{\alpha-2}(1-t)^{\beta-1} + (\beta - 1)t^{\alpha-2}(1-t)^{\beta-2}$

5. If $F(x) = f(g(x))$, where $f(-2) = 2, f'(-2) = 4, f'(5) = 3, g(5) = -2$, and $g'(5) = 5$, find $F'(5)$.

(a) -6
(b) -8
(c) 20
(d) 10
(e) 8

6. Assuming $x > 0$ and $y > 0$, which of the following expressions is equivalent to $\sin(\tan^{-1}(y/x))$:

(a) $\dfrac{x}{\sqrt{x^2+y^2}}$
(b) $\dfrac{x}{\sqrt{x^2-y^2}}$
(c) $\dfrac{y}{\sqrt{x^2+y^2}}$
(d) $\dfrac{y}{\sqrt{x^2-y^2}}$
(e) $\dfrac{x}{\sqrt{y^2-x^2}}$

7. Suppose $f'(x) \leq 0.5$ for all $x \in [-0.5, 0.5]$. If $f(-0.5) = 0.5$, what is the largest $f(0.5)$ can be.

(a) -1
(b) -0.5
(c) 0
(d) 0.5
(e) 1

8. Find the indefinite integral.

$$\int \sin^3(x)\cos(x)\,dx.$$

(a) $\sin^4(x)\cos(x)/4 + C$
(b) $\sin^3(x)\tan(x) + C$
(c) $\sin^2(x)\cos^2(x)$
(d) $\sin^3(x)/3 + C$
(e) $\sin^4(x)/4 + C$

9. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{3n}}$$

(a) $\frac{1}{9}$

(b) $\frac{1}{10}$

(c) $\frac{1}{11}$

(d) $\frac{1}{12}$

(e) $\frac{1}{13}$

10. Find the limit

$$\lim_{x \to 0^+} \frac{e^{1/x}}{\ln(x)}.$$

(a) 0

(b) $-\infty$

(c) ∞

(d) 1

(e) e

11. Find the slope of the tangent line to the following curve at the point $P(2, 8)$:

$$x = \sqrt{t}, \qquad y = t^2 - 2t$$

(a) 24

(b) 4^{-1}

(c) 6

(d) -24

(e) 4

12. If $\lim_{x \to a}[f(x) + g(x)] = A$ and $\lim_{x \to a}[f(x) - g(x)] = B$, find $\lim_{x \to a}[f(x)g(x)]$.

(a) $\frac{A^2 + B^2}{4}$

(b) $\frac{B^2 - A^2}{4}$

(c) $\frac{A^2 - B^2}{4}$

(d) $\frac{A^2 - B^2}{2}$

(e) $\frac{B^2 - A^2}{2}$

13. Find the equation of the tangent line to the curve $x^2 + xy + y^2 = 1$ when $x = 1$ and y is negative.

(a) $y = x - 3$
(b) $y = -2x$
(c) $y = x - 2$
(d) $y = -x + 2$
(e) $y = -x - 2$

14. Find an interval on which function f is increasing

$$f(x) = \ln(1 - x^2).$$

(a) $(1/2, 1)$
(b) $(0, 1)$
(c) $(-1/2, 1/2)$
(d) $(-1, 1/2)$
(e) $(-1, 0)$

15. Evaluate

$$\lim_{x \to -\infty} \left(\sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1} \right)$$

(a) $\dfrac{-5\sqrt{3}}{6}$
(b) $\dfrac{-5\sqrt{2}}{6}$
(c) $-\sqrt{3}$
(d) $\dfrac{5\sqrt{3}}{6}$
(e) $\sqrt{3}$

16. Find a vector perpendicular to the plane through the points $A(0, 1, 0)$, $B(10, 2, -3)$, and $C(8, 3, -2)$.

(a) $\langle -1, 1, 3 \rangle$
(b) $\langle 1, 1, -3 \rangle$
(c) $\langle -1, -1, -3 \rangle$
(d) $\langle 1, -1, 3 \rangle$
(e) $\langle 3, 1, -1 \rangle$

17. Solve the equation

$$\sin(2x) + \cos(2x) = 1.$$

(a) $x = 2k\pi$ or $x = \pi/3 + k\pi$, k an integer
(b) $x = k\pi$ or $x = \pi/2 + k\pi$, k an integer
(c) $x = k\pi$ or $x = \pi/4 + k\pi$, k an integer
(d) $x = 2k\pi$, k an integer
(e) $x = \pi/4 + k\pi$, k an integer

18. There exists one value of c such that the function f is continuous at all real numbers where

$$f(x) = \begin{cases} \dfrac{2\pi^3 - (2\pi^2 + c\pi)x + cx^2}{2\pi^3 - 3\pi^2 x + \pi x^2} & \text{for } x < \pi \\ \sin\left(\dfrac{\pi}{x}\right) & \text{for } x \geq \pi \end{cases}$$

Find the closest approximation to c.

(a) $c \approx 3.0000$
(b) $c \approx 3.1416$
(c) $c \approx 3.6396$
(d) $c \approx 3.9432$
(e) $c \approx 4.0000$

19. On which of the below intervals is the function f decreasing and concave up

$$f(x) = \frac{x}{x^2 - 4}.$$

(a) $(2, \infty)$
(b) $(-2, 2)$
(c) $(-2, \infty)$
(d) $(-\infty, -2)$
(e) $(-\infty, 2]$

20. A sequence $\{a_n\}$ is given by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$. Find $\lim_{n\to\infty} a_n$.

(a) -1
(b) -2
(c) 0
(d) 1
(e) 2

21. Suppose that the function f is the inverse of g and that $f(2) = 1$, $f'(1) = 2$, $f'(2) = 4$, and $g(2) = 1$. What is $g'(1)$?

(a) 1
(b) 4
(c) 1/4
(d) 2
(e) 1/2

22. Find the volume of the solid bounded by $y = x^3$, $y = 8$, and $x = 0$ rotated about the line $x = 3$.

(a) $\frac{260\pi}{5}$
(b) $\frac{264\pi}{5}$
(c) $\frac{246\pi}{5}$
(d) $\frac{221\pi}{5}$
(e) $\frac{256\pi}{5}$

23. The constant term in the expansion of $2x^3\left(mx^2 + \frac{2}{x}\right)^6$ is 768. Find m.

(a) $m = 2$
(b) $m = -2$
(c) $m = 1$
(d) $m = -1$
(e) $m = 3$

24. A spring has a natural length of 1 meter. A force of 25 Newtons stretches the spring by 0.25 meter. Determine how much work is done by stretching the spring from a length of 1.5 meters to 2.5 meters.

(a) 50 J
(b) 100 J
(c) 150 J
(d) 200 J
(e) 250 J

25. Based on the $\delta - \epsilon$ definition of limit, find the largest value of δ for the following limit and ϵ:

$$\lim_{x\to 2}(6 - 4x) = -2, \quad \epsilon = 0.1.$$

(a) 0.1
(b) 0.05
(c) 0.025
(d) 0.4
(e) 0.5

26. Find a value of c so that the function f is differentiable at all real numbers.

$$f(x) = \begin{cases} c(4^x - 4^4) & \text{for } x < 4 \\ \log_4(x) - 1 & \text{for } x \geq 4 \end{cases}$$

(a) $c = \ln 4$

(b) $c = \dfrac{1}{4^5[\ln 4]^2}$

(c) $c = \dfrac{[\ln 4]^2}{4^5}$

(d) $c = \dfrac{4^5}{[\ln 4]^2}$

(e) $c = 4$

27. Find the area of the region enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

(a) 18

(b) 9

(c) 4

(d) 8

(e) 2.5

28. Evaluate the following integral

$$\int e^{4x}\sqrt{1 + e^{2x}}dx.$$

(a)

$$\frac{1}{5}(1 + e^{2x})^{2.5} - \frac{1}{3}(1 + e^{2x})^{1.5} + C$$

(b)

$$\frac{1}{5}(1 + e^{2x})^{2.5} + \frac{1}{3}(1 + e^{2x})^{1.5} + C$$

(c)

$$\frac{1}{3}(1 + e^{2x})^{2.5} - \frac{1}{5}(1 + e^{2x})^{1.5} + C$$

(d)

$$\frac{1}{3}(1 + e^{2x})^{2.5} + \frac{1}{5}(1 + e^{2x})^{1.5} + C$$

(e)

$$\frac{1}{5}(1 + e^{x})^{2.5} + \frac{1}{3}(1 + e^{2x})^{1.5} + C$$

29. Note that $f(x) = x + e^{2x}$ is a one-to-one function. Find $f^{-1}(2)$ accurate to 4 decimal places.

(a) 56.5982
(b) 0.0177
(c) 0.2731
(d) 8.3891
(e) 70.3763

30. The volume of a growing spherical cell with radius r is given by $V = \frac{4}{3}\pi r^3$. Assume that the radius grows at a rate of 3 cm per minute. Find the rate of change for the volume when the radius is 10 cm.

(a) $40\pi\ cm^3$ per minute
(b) $120\pi\ cm^3$ per minute
(c) $400\pi\ cm^3$ per minute
(d) $1200\pi\ cm^3$ per minute
(e) none of the above

31. Find the area of the inner loop of $r = 2 + 4\cos\theta$ in polar coordinates.

(a) $3\pi - 4\sqrt{3}$
(b) $4\pi + 6\sqrt{3}$
(c) $3\pi + 4\sqrt{3}$
(d) $5\pi + 5\sqrt{3}$
(e) $4\pi - 6\sqrt{3}$

32. Suppose f is a function that satisfies the equation

$$f(x+y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y. Suppose also that

$$\lim_{x\to 0} \frac{f(x)}{x} = 1.$$

Find $f'(x)$.

(a) $f'(x) = x^2 - 1$
(b) $f'(x) = 1 - x^2$
(c) $f'(x) = 1 + x^2$
(d) $f'(x) = 1$
(e) $f'(x) = 0$

33. Find all anti-derivatives of f (use C to denote an arbitrary constant).

$$f(x) = \frac{1}{x^2(x-2)}.$$

(a) $\frac{1}{2x} + \ln \sqrt[4]{\left|1 - \frac{2}{x}\right|} + C^3$

(b) $\frac{1}{4x} + \ln \sqrt{\left|1 - \frac{2}{x}\right|} + C^2$

(c) $\frac{1}{2x} - \ln \sqrt[4]{\left|1 - \frac{2}{x}\right|} + C^5$

(d) $\frac{1}{2x} + \ln \sqrt[4]{\left|1 - \frac{2}{x}\right|} + C^2$

(e) $\frac{1}{4x} + \ln \sqrt{\left|1 - \frac{2}{x}\right|} + C$

34. Find $\lim_{x \to 0^+} (1 + \sin x)^{\cot x}$.

(a) 0

(b) 1

(c) e

(d) e^4

(e) does not exist.

35. A boy found a bicycle lock for which the combination was unknown. The correct combination is a four-digit number, $d_1 d_2 d_3 d_4$ where d_i, $i = 1, 2, 3, 4$ is selected from $1, 2, 3, 4, 5, 6$. Assume that the boy knows that the first digit is an even number and the last digit is 1 or 6. How many different lock combinations are possible with such a lock?

(a) 6^4

(b) 4^6

(c) 6^3

(d) 3^6

(e) 6^2

36. If $f'(x)$ is continuous, $f(2) = 0$ and $f'(2) = 7$, evaluate

$$\lim_{x \to 0} \frac{f(2+3x) + f(2+5x)}{x}$$

(a) 56

(b) 14

(c) 0

(d) 7

(e) 42

37. If $x\sin(\pi x) = \int_0^{x^2} f(t)dt$, where f is a continuous function, find $f(4)$.

(a) $-\frac{\pi}{2}$
(b) -1
(c) 0
(d) 1
(e) $\frac{\pi}{2}$

38. Find the $x-$coordinate of the point on the parabola $y = 1 - x^2$ at which the tangent line cuts from the first quadrant the triangle with the smallest area.

(a) $\frac{\sqrt{2}}{2}$
(b) $\frac{\sqrt{3}}{3}$
(c) $\frac{\sqrt{4}}{4}$
(d) $\frac{\sqrt{5}}{5}$
(e) $\frac{\sqrt{6}}{6}$

39. Suppose that Andy and Sam are playing a tennis match in which the first player to win four sets wins the match. How many different orders are possible (e.g. ASSAAA means Andy wins in six sets) if the match goes for 6 sets?

(a) 2
(b) 8
(c) 40
(d) 20
(e) none of the above

40. Find the interval of convergence I and the radius of convergence R for

$$\sum_{n=2016}^{\infty} \frac{(x-2)^n}{n3^n}.$$

(a) $I = [-1, 5)$, $R = 3$
(b) $I = [0, 4]$, $R = 2$
(c) $I = (1, 3)$, $R = 1$
(d) $I = (-\infty, \infty)$, $R = \infty$
(e) $I = \{2\}$, $R = 0$