THIRTY-SECOND ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

CHALLENGE EXAM 1988

Comprised of questions from the 1988 TMTA Geometry, Advanced Topics I and Advanced Topics II and questions submitted by Horace Williams, Vanderbilt University

Scoring formula: 4R - W + 40

Edited by: Larry Bouldin, Roane State Community College

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem there are listed 5 possible answers. You are to work each problem, determine the best answer, and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all questions. Do your very best on the questions you feel you know how to work. You will be penalized for incorrect answers, so it is advisable not to do wild guessing.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will be able to keep this booklet after the test is completed.

When told to do so, open your test booklet to page 2 and begin. When you have finished one page, go on to the next. The working time for the entire test is 80 minutes.

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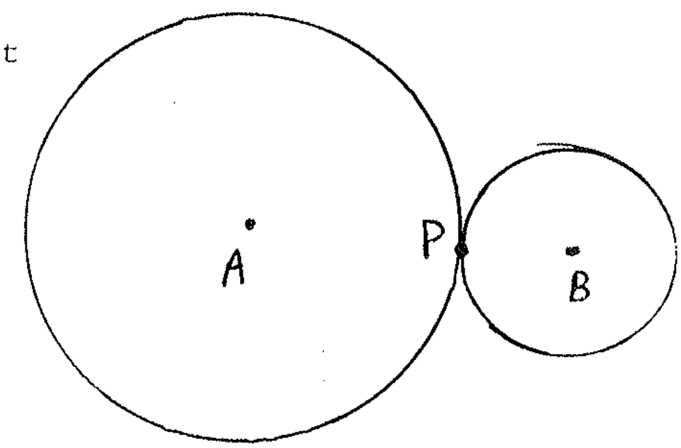
CHALLENGE EXAM

1. In order to determine the number of diagonals in a convex n-gon a student calculates the combination of n things (vertices) taken two at a time. She finds, however, that this calculation exceeds the actual number of diagonals in all cases by:

(a) 2 (b) n! (c) 2n (d) n (e) $\frac{n(n-3)}{2}$

2. In the figure at the right assume that circle A is immovable and that circle B can roll around the circumference of A. If the radius of circle B is half that of A and it makes one trip around circle A returning to its original position, how many rotations will it make?



(a) 1

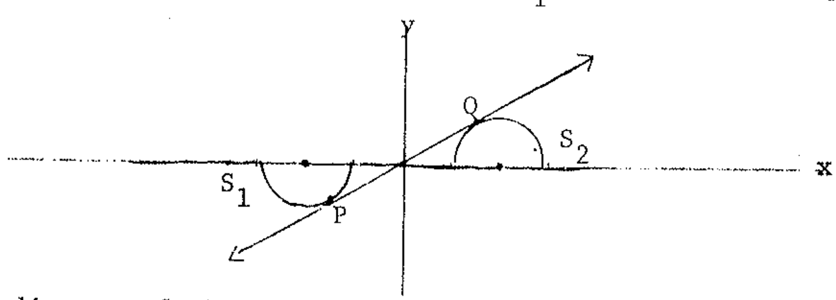
(b) 2

(c) 3

(d) 4

(e) $\frac{3\pi\sqrt{2}}{2}$

3. The semicircles S_1 and S_2 each have radius 1 and centers at the points (-2,0) and (2,0), respectively. (See Figure below,) There is a unique line $\overleftrightarrow{PQ}$ that is tangent to S_1 at P and is tangent to S_2 at Q.



The coordinates of the point Q are

(a) $(\sqrt{3}/2, 3/2)$ (b) $(1, \sqrt{3})$ (c) $(\sqrt{3}, 1)$

(d) $(3/2, \sqrt{3}/2)$ (e) $(3/2, \sqrt{3})$

4. A basketball tournament has eight teams and a team is out of the tournament if it loses two games. The minimum number of games needed to determine a winner is

(a) 11 (b) 12 (c) 13 (d) 14 (e) 15

5. If $y = (e^{2x} - e^{-2x})/2$ then $x =$

(a) $\frac{1}{2}\ell n(y + \sqrt{y^2 - 1})$ (b) $\frac{1}{2}\ell n(y - \sqrt{y^2 + 1})$ (c) $\frac{1}{2}\ell n(y + \sqrt{y^2 + 1})$

(d) $\ell n(y + \sqrt{y^2 - 1})$ (e) $\ell n(y - \sqrt{y^2 + 1})$

6. Given $F'(a)$ exists, which of the following are true?

1) $F'(a) = \lim_{h \to a} \frac{F(h) - F(a)}{h - a}$
2) $F'(a) = \lim_{h \to 0} \frac{F(a) - F(a-h)}{h}$
3) $F'(a) = \lim_{h \to 0} \frac{F(a+2h) - F(a)}{h}$
4) $F'(a) = \lim_{h \to 0} \frac{F(a+2H) - F(a+h)}{2h}$
5) $F'(a) = \lim_{h \to a} \frac{F(a+h) - F(a)}{h - a}$

a) Only (1) is TRUE

b) Only (1) and (2) are TRUE

c) Only (1) and (4) are TRUE

d) All are TRUE

e) Only (2) and (3) are TRUE

7. An isosceles triangle is inscribed in a circle of radius r. If the apex angle, θ, is restricted by $0 \leq \theta \leq \pi/2$, the largest value of the perimeter is:

a) $3r$

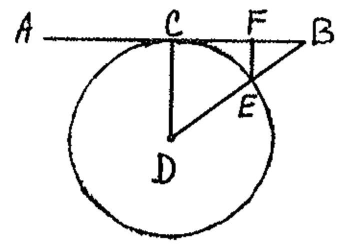
b) $3\sqrt{3}r$

c) $2(1 + \sqrt{2})r$

d) $4r$

e) $\sqrt{2}(1 + \sqrt{3})r$

8. Line segment $\overline{AB}$ has length 16 inches and is tangent to the circle centered at D whose diameter is 12 inches. The point of tangency C is the midpoint of $\overline{AB}$, and $\overline{FE}$ is parallel $\overline{CD}$. Find the length of $\overline{FB}$.

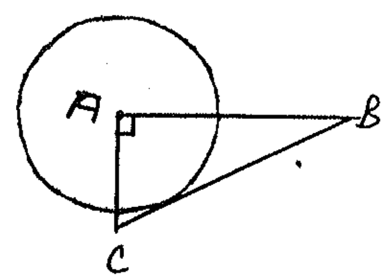


(a) 16/3
(b) 24/5
(c) 12/5
(d) 4
(e) 16/5

9. Three of four vertices of a parallelogram are at (-1,1), (2,-3/2) and (4,3) and the fourth is diagonally opposite (2,-3/2) Find the coordinates of the fourth vertex.

(a) (1,11/12)
(b) (1,11/2)
(c) (2,3/2)
(d) (1/2,-39/8)
(e) (1/2,35/8)

10. In the figure at the right, ABC is a right triangle and the circle centered at A is tangent to the hypotenuse $\overline{BC}$. Find the radius of the circle if $\overline{AB}$ = 2 and $\overline{AC}$ = 1.



(a) $2/\sqrt{5}$
(b) $\sqrt{2}$
(c) 1
(d) $\sqrt{2}/5$
(e) $\sqrt{5}$

11. For the system of linear equations

$$\begin{aligned} 3x + 4y - bz &= 17 \\ ax - ay + az &= -4 \\ bx - 3y + az &= -7 \end{aligned}$$

to have the solution (1,2,-3), the values of a and b should be, respectively,

(a) 0 and 0 (b) -2 and -1 (c) 1 and 3 (d) 1 and 2 (e) 1 and -5

12. If S_n is a convergent infinite sequence such that $S_{n+1} = \frac{1}{2} S_n + 1$ for all positive integers n, then $\lim_{n\to\infty} S_n$

(a) 0 (b) 1/2 (c) 1 (d) 3/2 (e) 2

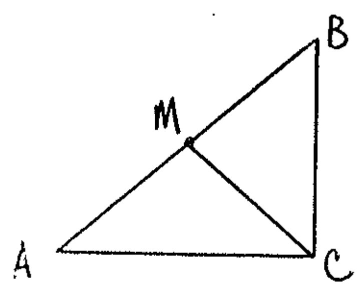
13. Which one of the following is a focus of the ellipse

$$\frac{x^2}{16} + \frac{(y+2)^2}{36} = 1 \quad ?$$

(a) $\left(2(1+\sqrt{5}),0\right)$
(b) $\left(0,-2(1+\sqrt{5})\right)$
(c) $\left(2(1+\sqrt{5}),-2(1+\sqrt{5})\right)$
(d) $(2\sqrt{5},-2\sqrt{5})$
(e) $(0,-2\sqrt{5})$

14. If $\angle BCA$ is a right angle, and M is the midpoint of side $\overline{AB}$, and the measure of $\angle CBA$ is 21°, then the measure of $\angle BMC$ is:

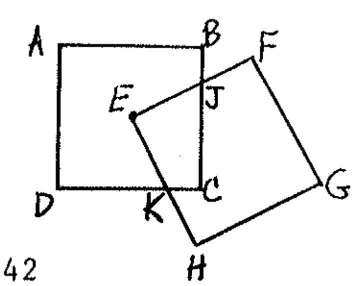
(a) 90°
(b) 69°
(c) 103°
(d) 138°
(e) 42°



15. The (perpendicular) distance between the lines y=x+1 and y=x is:

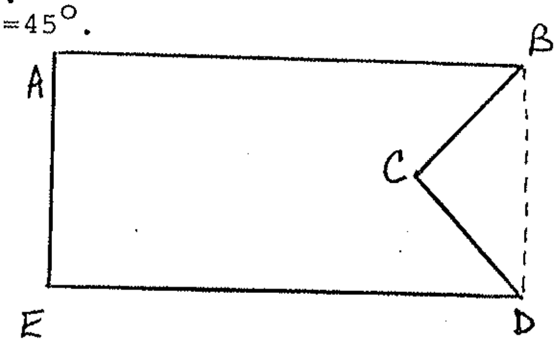
(a) 1
(b) $\sqrt{2}$
(c) 1/2
(d) $\sqrt{2}/2$
(e) $\sqrt{3}/2$

16. The two squares in the figure at the right each have side length 12 and are placed so that a corner of one lies at the center of the other. Find the area of quadrilateral EJCK if the length of BJ = 3.



(a) 32 (b) 48 (c) 36 (d) 24 (e) 42

17. In the figure m(∠A)=m(∠E)=m(∠DCB)=90°.
AE = 6", AB = 12" and m(∠CBD)=m(∠CDE)=45°.
What is the area of the figure ABCDE?



(a) 72 square inches
(b) 36 square inches
(c) 63 square inches
(d) 18 square inches
(e) 64 square inches

18 The ordered pair form of the complex number $z = x + yi$ is (x,y). If $z = 1 + i$, then the ordered pair form of $(3z-1)/(z-1)$ is

(a) $(-2,3)$ (b) $(-3,2)$ (c) $(1,1)$ (d) $(3,-2)$ (e) (∞,∞)

19. $[(a^{-1}+b^{-1})^{-1}+c^{-1}]^{-1}$ expressed as a single fraction without negative exponents is

(a) $\dfrac{ac+bc}{abc+a+b}$ (b) $\dfrac{a+b+c}{1}$ (c) $\dfrac{abc}{bc+ac+ab}$

(d) $\dfrac{1}{a+b+c}$ (e) $\dfrac{1+ac+bc}{a+b}$

20. If $1/x = \sqrt{2+\sqrt{2}}$, then which of the following is true?

(a) $2 + x^2 = 1/2x^2$ (b) $(2x^2 - 1)/x^2 = \sqrt{2}$ (c) $x = 1/(\sqrt{2} + \sqrt[4]{2})$

(d) $2x^4 - 4x^2 + 1 = 0$ (e) $2x^4 + 4x^2 + 1 = 0$

21. What is the largest positive value of k for which the graph of $y = 4x^2 + 2kx + 9$ lies on or above the x-axis?

(a) 2 (b) 4 (c) 6 (d) 10 (e) 12

22. If P and Q are points in the plane with polar coordinates $(-1,\pi/4)$ and $(3,-\pi/4)$, respectively, then the distance between P and Q is

(a) $\sqrt{10}$ (b) $\sqrt{64+\pi}/4$ (c) $\sqrt{64+\pi^2}/4$ (d) $\sqrt{256+\pi^2}/4$ (e) 4

23. The radius, r, of a right circular cone is increasing at a rate of 6 cm/min while its height h is decreasing at a rate of 15 cm/min. At the instant when $r = 50$ cm and $h = 50$ cm, the volume is:

a) Increasing at a rate of 2500 $\pi cm^3/min.$

b) Decreasing at a rate of 2500 $\pi cm^3/min.$

c) Decreasing at a rate of 7500 $\pi cm^3/min.$

d) Decreasing at a rate of 7500 $\pi cm^3/min.$

e) Unchanging.

24. For $c > 0, f(x) = \begin{cases} \dfrac{1}{|x|} & \text{if } |x| > c \\ a + bx^2 & \text{if } |x| \leq c \end{cases}$

If $f'(c)$ exists then:

a) $a = c$ and $b = \frac{-1}{c}$

b) $a = \frac{2}{c}$ and $b = \frac{-1}{c^3}$

c) $a = \frac{2}{3c}$ and $b = \frac{-1}{3c^3}$

d) $a = \frac{3}{2c}$ and $b = \frac{-1}{2c^3}$

e) a and b cannot be uniquely determined

25. $\lim_{x \to 0+} (\cos x)^{\frac{1}{\tan x}} =$

a) 1

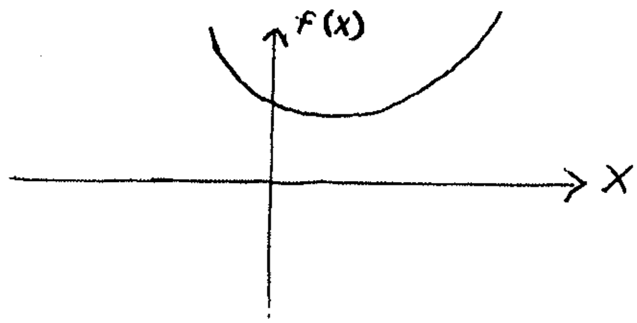
b) 0

c) ∞

d) -1

e) $\frac{\pi}{2}$

26. The curve drawn in the figure is the function $f(x) = Ax^2 + Bx + C$. Let $D = B^2 - 4AC$, then



a) $A > 0$, $B > 0$, $C > 0$, $D < 0$

b) $A > 0$, $B < 0$, $C < 0$, $D > 0$

c) $A < 0$, $B > 0$, $C > 0$, $D > 0$

d) $A > 0$, $B < 0$, $C > 0$, $D < 0$

e) $A < 0$, $B > 0$, $C < 0$, $D > 0$

27. The slope of the normal line to the curve $y = \ln[\cos^2 3x]$ at the point (0,0) is

a) 0

b) -6

c) $\frac{1}{6}$

d) $-\frac{1}{6}$

e) Undefined

28. How many terms are there in the expansion of $(a + b + c + d)^4$?

a) 4

b) 6

c) 28

d) 35

e) 64

29. Find the polynomial $f(x)$ of the fourth degree such that

$$f(0) = f(1) = 1, \;\; f'(0) = f''(0), \;\; f'''(1) = 54$$

a) $4x^4 - 4x^3 + 1$

b) $-x^4 + x^3 + 1$

c) $5x^4 - 5x^3 + 1$

d) $3x^4 - 3x^2 + 1$

e) $3x^4 - 3x^3 + 1$

30. Given two roots of the equation $2x^4 + x^3 + 5x^2 + 4x - 12 = 0$ are purely imaginary numbers. The other two roots must satisfy the equation

a) $2x^2 - x - 4 = 0$

b) $2x^2 - 2x + 1 = 0$

c) $2x^2 + x - 3 = 0$

d) $2x^2 + 3x - 2 = 0$

e) $2x^2 + 3x - 1 = 0$

31. The point on the graph of the function $f(x) = \sqrt{8x}$ which is nearest (3,0) is

a) (0,0)

b) $(1, 2\sqrt{2})$

c) $(\frac{1}{2}, 2)$

d) $(\frac{1}{8}, 1)$

e) $(\frac{7}{2}, 2\sqrt{7})$

32. $\lim_{x \to -\infty} (x + \sqrt{x^2 + 3x}) =$

a) $-\frac{3}{2}$

b) $\frac{3}{2}$

c) 0

d) ∞

e) $-\infty$

33. The region bounded by the curve $y = ln(x)$, the x axis, and the line $x = e^2$ is revolved about the y axis. The volume of the solid generated is

a) $2\pi e^4$

b) $\frac{1}{2}\pi e^4$

c) $\frac{\pi}{2}(3e^4 + 1)$

d) $2\pi(e^4 - 3)$

e) 0

34. What is the slope m of the line tangent to the curve $x^2 + 2y^2 + 4x - 8y = 24$ at the point where the curve crosses the positive y-axis?

a) m = -2/3

b) m = -1/4

c) m = -3/2

d) slope undefined

e) 1/2

35. If A and B are square matrices of order n and X^t denotes the transpose of matrix X then $(AB)^t$ is

a) A^tB^t

b) B^tA^t

c) $\frac{1}{2}(A^tB + AB^t)$

d) $\frac{1}{2}(AB + BA)^t$

e) $(BA)^t$

36. If three equal circles are tangent to each other, each to each, and enclose a space between the three areas equal to 200 ft^2, find the radius of each circle.

a) 322.7 ft.

b) 498.06 ft.

c) $\sqrt{200}$ ft.

d) 413.66 ft.

e) 512.4 ft.

37. A man sold a coat for $144. The number of dollars the coat cost was the same as the number of percent profit. What did the coat cost?

a) $80

b) $72

c) $102

d) $52

e) $70

38. A circle is 330 inches in diameter. How many points may be placed within this circle such that no two points shall be within 16 inches of each other, and no point within 5 inches of the circumference of the circle?

a) 328 points

b) 255 points

c) 620 points

d) 441 points

e) 367 points

39. I am now twice as old as you were when I was your age. When you are as old as I am now, the sum of our ages will be 100. What is your age?

a) 45

b) $24\frac{1}{4}$

c) $66\frac{2}{3}$

d) $33\frac{1}{3}$

e) 40

40. Johnson's cat went up a tree,
which was sixty feet and three;
Every day she climbed eleven,
Every night she came down seven.
Tell me, if she did not drop,
When her paws would touch the top.

a) 8 days

b) 10 days

c) 14 days

d) 19 days

e) 5 days