

TWENTY-SIXTH ANNUAL MATHEMATICS CONTEST
Sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

ADVANCED TOPICS TEST 1982

Edited by: The University of
Tennessee at Martin

Scoring Formula: $4R - W + 40$

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5. Let f be the function from the real numbers to the real numbers such that $f(x) = \frac{1}{3} \cdot x^3 - \frac{1}{2} \cdot x^2 - 6x$. The point on the graph of f where the slope of the tangent line has its least value is
- a) $(-2, \frac{22}{3})$ b) $(3, \frac{-27}{2})$ c) $(0, 0)$ d) $(\frac{1}{2}, \frac{-25}{4})$ e) $(\frac{1}{2}, \frac{-37}{12})$
6. Suppose that each of a, b, c, d is a real number and that $ad - bc = 2$. The inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is
- a) $\begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$ c) $\begin{bmatrix} 2d & -2b \\ -2c & 2a \end{bmatrix}$ e) $\begin{bmatrix} -a & c \\ b & -d \end{bmatrix}$
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7. Ten fair coins are tossed and the number of heads recorded. This procedure is done 1000 times, resulting in a list of 1000 numbers. The average of these numbers is likely to be near
- a) 5 b) 500 c) 100 d) 2.5 e) none of these
8. Consider a coordinate plane. Let R be the reflection of the plane in the x -axis. Let S be the reflection of the plane in the line $y = x$ that bisects quadrants I and III. Let T be the motion of the plane that consists of doing R , followed by S . Where does T send the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$?
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10. What is the median of this set of scores: {17, 20, 31, 22, 25}?

- a) 24
- b) 17
- c) 23
- d) 31
- e) 22

11. Consider these statements about the algebra of $n \times n$ matrices:

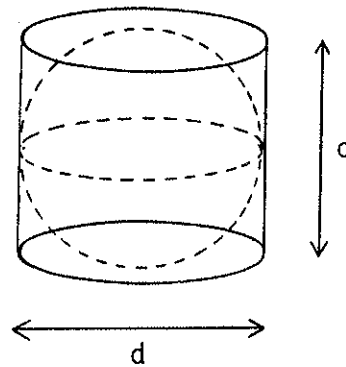
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The following is the set of true statements:

- a) {I, III}
- b) {I, II, III, IV}
- c) {II, III, IV}
- d) {II, IV}
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12. Consider the sphere inscribed inside a right circular cylinder. The ratio of the volume of the cylinder to the sphere is

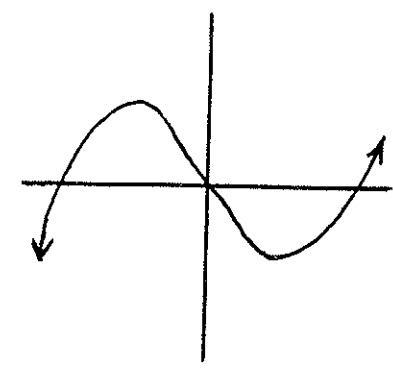
- a) 2:1
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- e) $\pi:2$



13. In a group of 4 randomly chosen people, the probability that at least two were born on the same day of the week is

- a) $\frac{4}{7}$
- b) $\frac{4}{2401}$
- c) $\frac{24}{5040}$
- d) $\frac{120}{343}$
- e) $\frac{223}{343}$

14. If the graph of $f(x)$ looks like this,

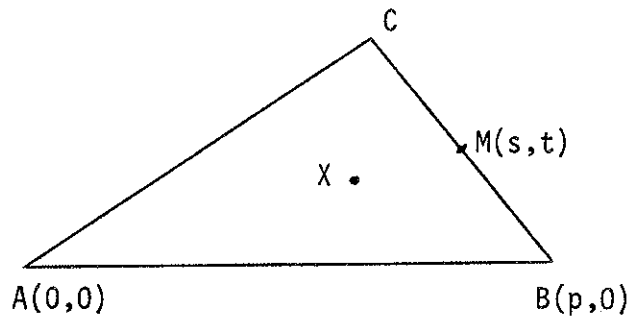


then the graph of $f'(x)$ looks like:

- a)
- b)
- c)
- d)
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15. The equation $f(x + y) = f(x) f(y)$ for all real numbers x and y is satisfied by
- a) $f(x) = 2^x$ c) $f(x) = 3^{2x}$ e) all of these
 b) $f(x) = 5^x$ d) $f(x) = 2^{3x}$
16. Urn I contains 5 red balls and 3 black balls. Urn II contains 4 red balls and 6 black balls. A fair coin is tossed twice. If HH occurs, then Urn I is chosen. Otherwise, Urn II is chosen. Then one ball is drawn from the chosen urn. If the ball that is drawn is red, what is the probability it comes from Urn II?
- a) $\frac{48}{73}$ c) $\frac{73}{160}$ e) $\frac{41}{80}$
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17. The height of a rocket ship t sec. after blast-off is h feet where $h = 500t^3 - 500t$. How fast is the ship moving 2 sec. after blast-off?
- a) 5500 ft./sec. c) 500 ft./sec. e) none of these
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18. Definition: If f is a function, its range is the set $\{y: y = f(x) \text{ for some } x \text{ in the domain of } f\}$. Let f be the function from the real numbers to the real numbers such that $f(x) = 4x + \frac{9}{x}$, $x > 0$. What is the range of f ?
- a) $\{y: y \geq 13\}$ c) $\{y: y \geq 12\}$ e) $\{y: 0 < y \leq 13\}$
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19. Let the determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 1$. The solution set of the system of three equations, $ax + by + cz = 0$, $dx + ey + fz = 0$, $gx + hy + iz = 0$, has _____ member(s). The correct number or phrase for the blank is
- a) no c) infinitely many e) none of the above
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20. Consider triangle ABC in the figure below. $A = (0, 0)$ and $B = (p, 0)$. $M = (s, t)$ is the mid-point of side BC. Let X be the point $(\frac{2}{3}s, \frac{2}{3}t)$.



- The line CX meets AB in a point Y. Which of the following statements is true about the first coordinate of Y?
- There is not enough information to say anything.
 - It equals $(s + t)/2$.
 - It equals $(s + t + p)/3$.
 - It equals $\frac{1}{3}p + (s + t)/2$.
 - It equals $p/2$.
21. If x is a real number other than zero, we can infer that
- $x^2 > x$
 - $x^n > x$ for sufficiently large n
 - $x^2 > x$ if $x > 0$
 - $x^2 < x$ if $x < 1$
 - none of the above
22. Which of the following arrangements of men comprise a chronological list?
- Riemann, Archimedes, Euclid, Newton
 - Archimedes, Riemann, Leibniz, Euclid
 - Euclid, Gauss, Leibniz, Newton
 - Dedekind, Gauss, Euclid, Newton
 - Pythagoras, Newton, Gauss, Dedekind

23. What is the greatest value assumed by the function $f(x) = 2 - |x - 2|$?
- a) 1
 - b) 2
 - c) 3
 - d) 4
 - e) none of the above
24. The solution set for the inequality $\frac{1}{x^2} > 1$ is
- a) $\{x|x < -1 \text{ or } x > 1\}$
 - b) $\{x|x < -1 \text{ and } x > 1\}$
 - c) $\{x|-1 < x < 1\}$
 - d) $\{x|-1 \leq x \leq 1\}$
 - e) none of the above
25. In any triangle, the point that is equidistant from the three vertices is the intersection of:
- a) the angle bisectors
 - b) the incircle and the circumcircle
 - c) the medians
 - d) the perpendicular bisectors of the sides
 - e) the symmedians
26. Tom could do a task in 3 hours. Dick could do the same task in 2 hours, and Sam in $1\frac{1}{2}$ hours. How long would it take Tom and Sam to do it working together?
- a) 2 hours
 - b) $2\frac{1}{6}$ hours
 - c) $\frac{2}{3}$ hours
 - d) 1 hour
 - e) none of the above
27. Find the value of K for which the following equation is an identity:
- $$(\cos x + \sin x)^2 + K \sin x \cos x - 1 = 0$$
- a) 1
 - b) -1
 - c) -2
 - d) 2
 - e) none of the above

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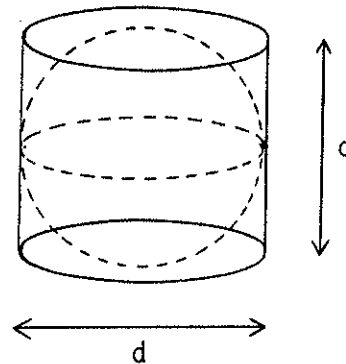
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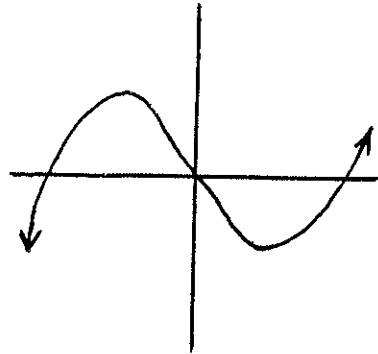
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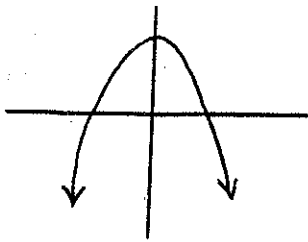
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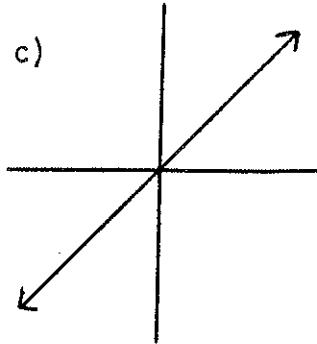


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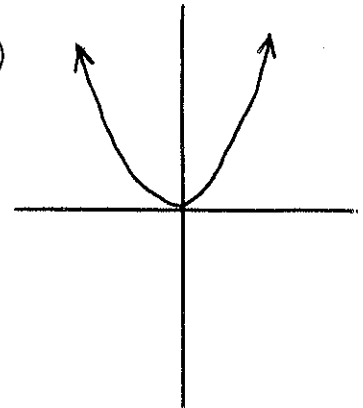
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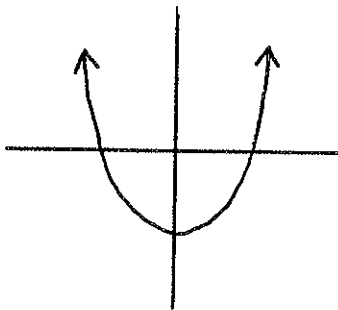
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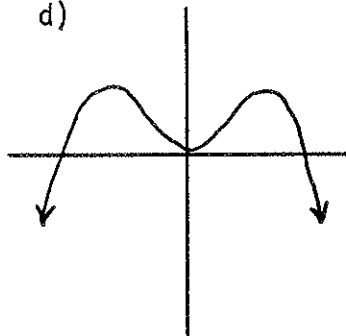
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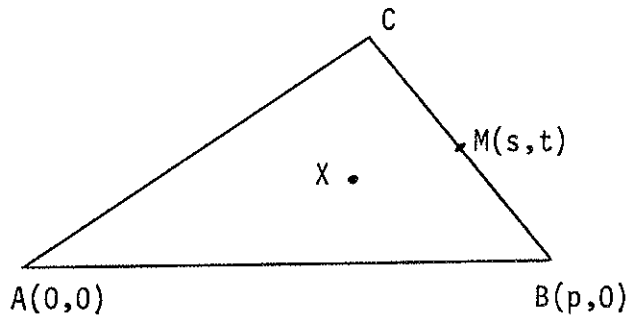


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