FORTY-NINTH ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Calculus and Advanced Topics 2005

Prepared by:

Department of Mathematics and Physics
Lambuth University
Jackson, Tennessee

Reviewed by:

Mathematics Faculty
Austin Peay State University
Clarksville, TN 37044

Coordinated by: Sara Hakim

Scoring formula: 4R - W + 40

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft head (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly 80 minutes to work.

Contributors to TMTA for the Annual Mathematics Contest:

Dr. Hal Ramer, President, Volunteer State Community College, Gallatin, Tennessee
Donnelley Printing Company, Gallatin, Tennessee
TRW Commercial Steering Division, Lebanon, Tennessee
Wright Industries, Inc., Nashville, Tennessee

Calculus and Advanced Topics Test 2005

1. The area between the curves $y = x^3 - x^2$ and $y = x^2 + x - 2$ is represented by

   a) $\int_{-1}^{2}(x^3 - 2x^2 - x - 2)dx$

   b) $\int_{-1}^{2}(-x^3 + 2x^2 + x - 2)dx$

   c) $\int_{-1}^{1}(x^3 - 2x^2 - x + 2)dx - \int_{1}^{2}(x^3 - 2x^2 - x + 2)dx$

   d) $\int_{-1}^{1}(-x^3 + 2x^2 + x - 2)dx + \int_{1}^{2} -2x^2 - x + 2 + x^3 dx$

   e) $\int_{-1}^{1}(x^3 - 2x^2 - x + 2)dx$

2. For what value of k does the graph of $f(x) = x^4 - \frac{4k}{3}x^3 - 2x^2 + 4kx + \frac{1}{3}$ have a relative minimum at x = 2?
   a) k=0
   b) k=1
   c) k=-2
   d) k=-1
   e) k=2

3. How many ways can the letters EXCELLENT be arranged?
   a) 30240
   b) 362880
   c) 181440
   d) 60480
   e) 720

4. $\lim_{x \to 0} \frac{x^2}{e^x - x - 1} =$
   a) ∞
   b) 2
   c) $-\infty$
   d) 0
   e) 1

5. The sum $\frac{8}{3}+\frac{16}{9}+\frac{32}{27}+\ldots$
   a) converges to 6
   b) converges to 3
   c) diverges
   d) converges to 8
   e) converges to 12

6. $\int_0^{\pi} \cos^2 x \sin^2 x dx =$
   a) π/4
   b) 0
   c) 4/15
   d) π/8
   e) –4/15

7. Find the following limit: $\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^{n\sqrt{2}}$
   a) $e^{-\frac{1}{2\sqrt{2}}}$
   b) $e^{\sqrt{2}}$
   c) $e^{2\sqrt{2}}$
   d) $e^{-2\sqrt{2}}$
   e) $e^{-\sqrt{2}}$

8. $\int_0^1 x^2 e^x dx$
   a) 2 - e
   b) 2e +1
   c) e - 2
   d) e + 2
   e) –e - 1

9. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{(i-1)^2}{n^3} =$

a) 1/6
b) ¼
c) 1/3
d) 0
e) 1

10. The volume of the solid generated by revolving the region bounded by the graphs of the equations $y = \sqrt{x}$, y = 0, x = 4 about the y-axis is
a) 5/3 π
b) 3/5 π
c) 1280/3 π
d) 32/5 π
e) 128/5 π

11. A machine is rolling a metal cylinder under pressure. The radius of the cylinder is decreasing at a constant rate of 0.04 inches per second and the volume is 128π cubic inches. At what rate is the length changing when the radius is 1.5 inches?

a) 59.92 in/sec.
b) 3.034074 in/sec.
c) –3.034074 in/sec.
d) –6.8267 in/sec.
e) –59.92 in/sec.

12. Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius r.

a) $\frac{r}{\sqrt{2}}$ and $\sqrt{\frac{\sqrt{2}r^2 - r}{\sqrt{2}}}$

b) $\frac{r}{\sqrt{2}}$ and $\frac{r}{\sqrt{2}}$

c) $r\sqrt{2}$ and $r\sqrt{1-\sqrt{2}}$

d) $\frac{r}{\sqrt{2}}$ and $\frac{2r}{\sqrt{2}}$

e) $r\sqrt{2}$ and $\frac{2r}{\sqrt{2}}$

13. Given $x^2 + y^2 = 25$, the value of y'' at (-3,-4) is equal to

a) $\frac{25}{256}$

b) $\frac{-25}{256}$

c) $\frac{-17}{128}$

d) $\frac{-25}{64}$

e) $\frac{25}{64}$

14. If $y = \left(\frac{x^2+1}{(x+1)\sqrt{x}}\right)^{-3}$, $\frac{dy}{dx} =$

a) $3\left(\frac{(x+1)\sqrt{x}}{x^2+1}\right)^2\left(\frac{(x^2+1)\sqrt{x}+(x+1)\frac{1}{2\sqrt{x}}-2x(x+1)\sqrt{x}}{(x^2+1)^2}\right)$

b) $-3\left(\frac{(x+1)\sqrt{x}}{x^2+1}\right)^3\left(\frac{2x}{x^2+1}-\frac{1}{x+1}-\frac{1}{2x}\right)$

c) $3\left(\frac{(x+1)\sqrt{x}}{x^2+1}\right)^2\left(\frac{(x^2+1)(\sqrt{x}+(x+1)\frac{1}{2\sqrt{x}})}{(x^2+1)^2}\right)$

d) $-3\left(\frac{(x+1)\sqrt{x}}{x^2+1}\right)^{-2}4x\sqrt{x}$

e) $-3\left(\frac{x^2+1}{(x+1)\sqrt{x}}\right)^{-2}\left(\frac{(x+1)2x\sqrt{x}-(x^2+1)(\frac{3}{2}\sqrt{x}+\frac{1}{2\sqrt{x}})}{(x+1)^2x^2}\right)$

15. Rolle's theorem may not be applied to which function?

a) $f(x) = x^2 - 3x + 2$ on [1,2]
b) $f(x) = x^2 - 3x + 2$ on [0,3]
c) $f(x) = |x|$ on [-1,1]
d) $f(x) = \frac{x^2 - 2x - 3}{x + 2}$ on [-1,3]
e) $f(x) = \sqrt{9 - x^2}$ on [-2,2]

16. If A is a 3x3 matrix and det(A)=6, $\det(2A)^{-2} =$

a) $\frac{1}{2304}$
b) $\frac{1}{72}$
c) $\frac{1}{24}$
d) $\frac{1}{12}$
e) $\frac{1}{48}$

17. Find the area of the region bounded by the graphs of $y = x - 4$ and $y^2 = 2x - 5$.

a) 20 square units
b) $\frac{20}{3}$ square units
c) 16 square units
d) $\frac{16}{3}$ square units
e) $\frac{25}{3}$ square units

18. How many five letter 'words' can be made if the first and last letters must be vowels and letters may not be repeated?

a) 242,414
b) 242880
c) 159600
d) 438480
e) 185220

19. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}}{x+1} =$

a) 0
b) 1
c) $+\infty$
d) ½
e) ¼

20. Let $u_n > 0$. If $\sum\limits_{n=0}^{\infty} u_n$ converges, then

a) $\sum\limits_{n=0}^{\infty} \sqrt{u_n}$ converges
b) $\sum\limits_{n=0}^{\infty} \sqrt{u_n}$ diverges
c) $\sum\limits_{n=0}^{\infty} \dfrac{\sqrt{u_n}}{n}$ converges
d) $\sum\limits_{n=0}^{\infty} \dfrac{\sqrt{u_n}}{n}$ diverges
e) none of the above.

21. Solve the ODE $\dfrac{dx}{dt} = \dfrac{e^t + x}{2\sin x - t}$

a) $xe^t + t + 2\sin x = C$
b) $e^t + xt - 2\cos x = C$
c) $te^t + x + 2\sin x = C$
d) $e^t + xt + 2\cos x = C$
e) $e^t + xt + 2\sin x = C$

22. $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}} =$

a) 2
b) 3
c) 4
d) 5
e) 6

23. Let $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & x^2+y^2\neq 0\\ 0 & x^2+y^2=0\end{cases}$. Which of the following is incorrect?

a) $f_x(0,0)=0$
b) $f_y(0,0)=0$
c) $f(x,y)$ is continuous at (0,0)
d) $f(x,y)$ is not continuous at (0,0)
e) none of the above.

24. Suppose f is the inverse of g, g(2)=8, $f'(2)=8$, $f'(8)=2$ and f(2)=3. What is $g'(2)$?

a) 1
b) $\frac{1}{2}$
c) $\frac{1}{8}$
d) $\frac{1}{16}$
e) -21

25. Suppose u = g(x) is differentiable at x = 1 and y = f(u) is differentiable at u = g(1). If $y=(f\circ g)(x)$ has a horizontal tangent at x=1, which of the following can you conclude?

a) The tangent to the graph of g at x=1 is horizontal.
b) The tangent to the graph of f at g(1) is horizontal
c) The tangents to the graph of f at g(1) and g at x=1 are horizontal
d) The tangent to the graph of f at g(1) or the tangent to the graph of g at x=1 is horizontal.
e) None of the above.

26. $f(x)=1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+...+\frac{x^n}{(n+1)!}+...$ is the McLaurin series for

a) $\frac{e^{-x}}{x}$
b) e^x
c) $1+\ln(1+x)$
d) $\frac{e^x-1}{x}$
e) $\frac{e^x}{x}$

27. Suppose $r_1=1-\cos^2\theta$, $0<\theta<\pi$, L_1 is the length of r_1, $r_2=2\sin^2\theta$, $0<\theta<\pi$, and L_2 is the length of r_2. Which of the following is true?

a) $L_1=\frac{1}{2}L_2$
b) $L_1=\sqrt{2}L_2$
c) $L_1=\frac{1}{\sqrt{2}}L_2$
d) $L_1=2L_2$
e) $L_1=\frac{1}{4}L_2$

28. Find the interval on which $y=f(x)=\int_0^x \frac{1}{1+t+2t^2}dt$ is concave up.

a) $\left(-\frac{1}{2},\infty\right)$ b) $\left(-\infty,\frac{1}{4}\right)$ c) $\left(\frac{1}{4},\infty\right)$ d) $\left(-\frac{1}{4},\infty\right)$ e) $\left(-\infty,-\frac{1}{4}\right)$

29. $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}=$

a) 0
b) ½
c) 1
d) ∞
e) 1/3

30. $\prod_{j=2}^{n}(1-\frac{1}{j^2}) =$

a) ½

b) $\frac{1}{2}(1+n^2)$

c) $1-\frac{1}{n^2}$

d) $\frac{n^2-1}{2n^2}$

e) $\frac{n+1}{2n}$

31. $\int\sqrt{2x^2-1}dx$

a) $\frac{1}{2}\left(x\sqrt{x^2-1}+\ln(x+\sqrt{x^2-1})\right)+C$

b) $\frac{1}{2}\left(x\sqrt{x^2-1}-\ln(x+\sqrt{x^2-1})\right)+C$

c) $\frac{1}{2\sqrt{2}}\left(x\sqrt{2x^2-1}+\ln(x+\sqrt{2x^2-1})\right)+C$

d) $\frac{1}{2\sqrt{2}}\left(x\sqrt{2x^2-1}-\ln(x+\sqrt{2x^2-1})\right)+C$

e) $\frac{1}{2\sqrt{2}}\left(x\sqrt{2}\sqrt{2x^2-1}-\ln(x\sqrt{2}+\sqrt{2x^2-1})\right)+C$

32. Consider $\sum_{n=1}^{\infty}\frac{(-1)^n\alpha^n}{n^k}$ ($\alpha>0$, $k>0$). Which of the following must be true?

a) If $\alpha > 1$, the series converges absolutely for $k > 1$, but converges conditionally for $0 < k < 1$.

b) If $\alpha < 1$, the series converges absolutely for $k > 1$, but converges conditionally for $0 < k < 1$.

c) If $\alpha > 1$, the series converges conditionally for $k > 1$, but diverges for $0 < k < 1$.

d) If $\alpha < 1$, the series converges absolutely for $k > 0$.

e) None of the above.

33. Find the equation of the normal line to the curve defined by the equation $x^3y^4 - 5 = x^3 - x^2 + y$ at the point $(2,-1)$.

a) 33x + 4y – 62 = 0 b) 33x – 4y – 62 = 0 c) 33x + 4y + 62 = 0

d) 4x + 33y + 41 = 0 e) 4x + 33y – 41 = 0

34. $\sum_{n=1}^{\infty} \frac{2n-1}{2^n} =$

a) 1 b) 2 c) 3 d) 4 e) 5

35. $\int_{-\infty}^{\infty} e^{-x^2} dx =$

a) π
b) $\sqrt{\pi}$
c) $\pi/2$
d) π^2
e) $\frac{\pi^2}{2}$

36. If $f(x) = \sum_{i=1}^{n} (x - a_i)^2$, where $a_1, a_2, \ldots a_n$ are constants, the minimum value of f(x) occurs when x =

a) 0
b) max $\{a_1, a_2, \ldots a_n\}$
c) $\frac{a_1 + a_2 + \ldots + a_n}{n}$
d) $a_1 + a_2 + \ldots + a_n$
e) min $\{a_1, a_2, \ldots a_n\}$

37. Suppose $a_n = \cos n\pi$. If S_n is the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$,

a) S_n converges to 0
b) S_n converges to 1
c) S_n converges to –1
d) S_n diverges
e) S_n converges to π.

38. Suppose $n \geq 2$ and let $f_n(x) = \begin{cases} n^2 x & 0 \leq x \leq \frac{1}{n} \\ 2n - n^2 x & \frac{1}{n} < x \leq \frac{2}{n} \\ 0 & x > \frac{2}{n} \end{cases}$ Which of the following is true?

a) $\int_0^1 f_n(x)dx < \int_0^1 f_{n+1}(x)dx$
b) $\int_0^1 f_n(x)dx > \int_0^1 f_{n+1}(x)dx$
c) $\int_0^1 f_n(x)dx = \int_0^1 f_{n+1}(x)dx$.
d) $2\int_0^1 f_n(x)dx = \int_0^2 f_{n+1}(x)dx$
e) $1 + \int_0^1 f_n(x)dx = \int_0^1 f_{n+1}(x)dx$

39. A balloon is rising vertically above a lake at a constant rate of 1 ft./sec. When the balloon is 65feet above the ground, a swimmer passes under it travelling along a straight path at a constant rate of 17ft/sec. How fast is the distance between the swimmer and the balloon increasing after 3 seconds?

a) 11 ft/sec.
b) 12 ft./sec
c) 13 ft/sec
d) 14 ft./sec
e) 15 ft./sec

40. What is the area of the largest isosceles triangle that can be inscribed in a circle of radius r?

a) $\frac{r^2}{2}$
b) $\frac{3\sqrt{3}r^2}{4}$
c) $2r^2$
d) $\frac{3\sqrt{3}r^2}{2}$
e) $\sqrt{3}r^2$