FORTY-SIXTH ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Calculus and Advanced Topics 2002

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Scoring formula: 4R − W + 40

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft head (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly 80 minutes to work.

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Calculus and Advanced Topics

1. A line perpendicular to the line 3x + 5y − 7 = 0 and passing through the point (2,11) has an equation

   (a) -3(x-2) + 5(y-11) = 0 (b) 3(x-2) + 5(y-11) = 0 (c) 2(x-3) + 11(y-5) = 0

   (d) -5(x-2) + 3(y-11) = 0 (e) -11(x-3) + 2(y-5) = 0

2. The solution of | x + 5| = 2x −7 is

   (a) x = 12 and $x = \frac{2}{3}$ (b) $x = \frac{2}{3}$ (c) x = 7

   (d) x = -7 (e) x = 12

3. The number of unit vectors perpendicular to a given non-zero vector in R^3 is

   (a) 0 (b) 1 (c) 2 (d) 3 (e) infinite

4. Thirty people are at a meeting. If everyone shakes everyone else's hand exactly once, how many handshakes took place?

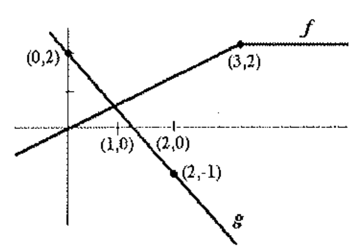
   (a) 60 (b) 30 (c) 435 (d) 870 (e) 900

5. If $f(x) = x^2 - 9$, $g(x) = 2x - 3$, then $g(f(x+3))$ equals

   (a) $2x^2 - 3$ (b) $4x^2 - 3$ (c) $2x^2 - 12x - 3$

   (d) $2x^2 - 21$ (e) $2x^2 + 12x - 3$

6. If the graphs of f and g are as pictured then $(fg)'(2)$ equals



   (a) −2 (b) −8/3 (c) −1 (d) −4/3

   (e) fg is not differentiable at 2.

7. If the equation of the normal line to the graph of a differentiable function at (2,5) is $2x + 3y - 19 = 0$. Find $f'(2)$.

(a) $-\frac{3}{2}$ (b) $-\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $\frac{2}{3}$

(e) $\frac{5}{2}$

8. Suppose that $f : R \to R$ is a differentiable function which is increasing. Then

(a) the slope of the tangent line increases as x increases

(b) the slope of the tangent line cannot be positive

(c) the slope of the tangent line decreases as x increases

(d) the slope of the tangent line cannot be negative

(e) the slope of the tangent line has to be zero for every x.

9. Suppose that two circles C_1, C_2 in the plane do not have points in common. Then

(a) there is exactly one line tangent to both C_1 and C_2.

(b) there are exactly two lines tangent to both C_1 and C_2.

(c) There are no lines tangent to both C_1 and C_2 or there are exactly two lines tangent to both C_1 and C_2.

(d) there are exactly three lines tangent to both C_1 and C_2.

(e) there are no lines tangent to C_1 and C_2 or there are exactly four lines tangent to both C_1 and C_2.

10. Find a vector parallel to the straight line defined by x = -5t + 2, y = -2t + 1, z = t – 4.

(a) <2, 1, -4> (b) <-3, -1,-3> (c) <2,1,4> (d) <-10,-4,2>

(e) <5,-2,1>

11. Given the matrix product $\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix} A \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} = \begin{bmatrix}1&2&3\\4&5&2\\3&2&1\end{bmatrix}$, find the matrix A.

(a) $\begin{bmatrix}1&2&3\\3/2&1&1/2\\2&5/2&1\end{bmatrix}$ (b) $\begin{bmatrix}1&2&3\\2&5/2&1\\3/2&1&1/2\end{bmatrix}$ (c) $\begin{bmatrix}1&3&2\\2&1&5/2\\1&1/3&2/3\end{bmatrix}$

(d) $\begin{bmatrix}1&3&2\\2&5/2&1\\3/2&1&1/2\end{bmatrix}$ (e) $\begin{bmatrix}1&2&3\\1&2/3&1/3\\2&5/2&1\end{bmatrix}$

12. If f(x) is differentiable function and g(x) = $f(2x^3 + 5x^2 + 7)$, what is $g'(x)$?

(a) $f'(6x^2 + 10x)$
(b) $f'(2x^3 + 5x^2 + 7)$
(c) $(2x^3 + 5x^2 + 7)\,f'(6x^2 + 10x)$
(d) $(6x^2 + 10x)\,f'(x)$
(e) $(6x^2 + 10x)\,f'(2x^3 + 5x^2 + 7)$

13. Evaluate the integral $\int \sin 2x \cos 2x \, dx$

(a) $\frac{\sin 4x}{8} + c$ (b) $\frac{\cos 4x}{8} + c$ (c) $\frac{-\cos 4x}{8} + c$

(d) $\frac{-\sin 4x}{4} + c$ (e) $\frac{-\cos 4x}{4} + c$

14. For any real numbers a and b define $a * b = a + b - a \bullet b$ where +, -, • are usual real number addition, subtraction, and multiplication operations. Find the sum of the solutions of the equation $(x * x) * x = x$.

(a) –2 (b) 2 (c) 3 (d) –3

(e) 0

15. If $f(x) = x^3 + 2x$ then $(f^{-1})'(12)$ equals

(a) $\frac{1}{2} + \frac{1}{6\sqrt[3]{18}}$ (b) $\frac{1}{434}$ (c) $\frac{1}{14}$ (d) $\sqrt{\frac{10}{3}}$

(e) 0

16. The polynomial $x^4 + ax^3 + bx^2 + cx + 10$ where a,b,and c real numbers is known to have 2 + i and 1 as zeroes. Which of the following must be a zero for the polynomial?

(a) 2 (b) 5 (c) –2 (d) –1 (e) –5

17. $\int_0^{\ln 2} \frac{1}{e^{-x} + 1}\, dx$ equals

(a) $1 + \ln 2$

(b) $\frac{\ln 2}{\ln 2 - \frac{1}{2}}$

(c) $\ln \frac{3}{4}$

(d) $\ln \frac{3}{2}$

(e) $\frac{2}{2 + \ln 2}$

18. An urn contains 25 marbles with 5 marbles each of five different colors. Three marbles are drawn without replacement. The probability that there are at least two of same color is

(a) $\frac{21}{46}$ (b) $\frac{2}{5}$ (c) $\frac{25}{46}$ (d) $\frac{10}{23}$

(e) $\frac{47}{198}$

19. If f is continuous and $\int_0^4 f(x)\,dx = 6$ and $\int_2^4 f(x)\,dx = 4$, evaluate $\int_0^2 x\,f(x^2)\,dx$

(a) 3 (b) 1 (c) $2\sqrt{2}$ (d) 12

(e) 8

20. $\lim_{x \to 1} (3x - 2)^{\frac{1}{x-1}}$ equals

(a) 3 (b) 20 (c) 1 (d) e^3

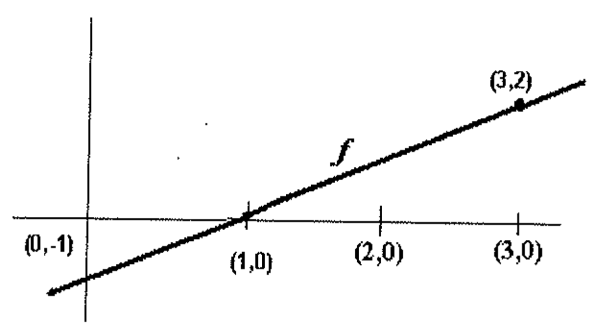
(e) ∞

21. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $B = \begin{bmatrix} 3a_{13} - a_{11} & a_{12} + 4a_{11} & 2a_{11} \\ 3a_{23} - a_{21} & a_{22} + 4a_{21} & 2a_{21} \\ 3a_{33} - a_{31} & a_{32} + 4a_{31} & 2a_{31} \end{bmatrix}$. If det A = 3 then

det B equals

(a) –72 (b) 72 (c) –18 (d) 18 (e) –24

22. If the graph of f is as pictured, the best estimate of $\int_0^3 f''(x)\,dx$ is



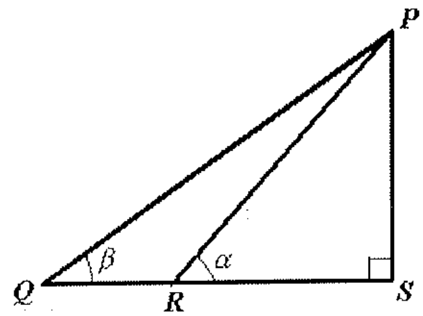
(a) $-\frac{1}{2}$ (b) $\frac{3}{2}$ (c) 0 (d) 3 (e) –2

23. Given fixed points A and B in the plane, the set of all points P such that |PA| = 2|PB| is

(a) a point (b) a line (c) a circle (d) a noncircular ellipse

(e) a hyperbola

24. In the diagram if QR = d, then PS equals



(a) $\dfrac{\sin\beta}{\sin(\alpha-\beta)}d$

(b) $\dfrac{\tan\beta}{\tan\alpha - \tan\beta}d$

(c) $\dfrac{d}{\tan\alpha - \tan\beta}$

(d) $\dfrac{\sin\alpha \, \sin\beta}{\cos\alpha\,(\sin\alpha - \sin\beta)}d$

(e) $\dfrac{d}{\cot\beta - \cot\alpha}$

25. Let $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c, d are real numbers not equal to zero. Then the graph of $y = f(x)$ intersects the x-axis

(a) at exactly one point.

(b) at exactly two distinct points.

(c) at least at one point.

(d) at most at one point.

(e) at exactly three different points.

26. The total length of two sides of a triangle is increased and the third side stays the same. Then the area of the triangle

(a) will always increase

(b) will always decrease

(c) will increase or decrease

(d) could increase, could decrease, or could stay the same

(e) will not change

27. Let a_n be an integer for n = 1,2,3,.... If $\lim_{n\to\infty} a_n$ exists, then

(a) $a_n < a_{n+1}$ for infinitely many values of n.

(b) $a_n > a_{n+1}$ for infinitely many values of n.

(c) $a_n = a_{n+1}$ for only finitely many values of n.

(d) $a_n = a_{n+1}$ for all but finitely many values of n.

(e) $a_n \neq a_{n+1}$ for infinitely many values of n.

28. Find the matrix A if the inverse of 2A is equal to $\begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}$.

(a) $\begin{bmatrix} 8 & -10 \\ 12 & -10 \end{bmatrix}$ (b) $\begin{bmatrix} 2.5 & 2 \\ 3 & 2.5 \end{bmatrix}$ (c) $\begin{bmatrix} 2.5 & -2 \\ -3 & 2.5 \end{bmatrix}$ (d) $\begin{bmatrix} 10 & -8 \\ -12 & 10 \end{bmatrix}$

(e) $\begin{bmatrix} -2.5 & 2 \\ 3 & -2.5 \end{bmatrix}$

29. The sum of the zeros of the polynomial $x^5 + 8x^4 - 7x^3 + 2x^2 + x + 3$ is equal to

(a) -7 (b) -8 (c) 8 (d) 7 (e) 3

30. Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

(a) $\frac{3}{2}$ (b) $\frac{3}{4}$ (c) 2 (d) 1 (e) $\frac{2}{3}$

31. If f(x) is an invertible function, and g(x) = 2 f(x) + 5, what is $g^{-1}(x)$?

(a) $2 f^{-1}(x) + 5$ (b) $2 f^{-1}(x) - 5$ (c) $\frac{1}{2f^{-1}(x)+5}$

(d) $\frac{1}{2} f^{-1}(x) - 5$ (e) $f^{-1}(\frac{x-5}{2})$

32. Find the equation of the tangent line to the parabola $y^2 = 5x$ which is parallel to the line $y = 4x + 1$.

(a) $4x - 4y - 35 = 0$ (b) $8x - 2y - 5 = 0$ (c) $4x - 4y + 35 = 0$

(d) $64x + 16y - 5 = 0$ (e) $64x - 16y + 5 = 0$

33. Compute $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$

(a) -1 (b) 2 (c) 3 (d) 6 (e) -3

34. Let $\sum_{n=1}^{\infty} a_n$ be a convergent infinite series of positive terms with sum s. Define $\{b_n\}$ as follows; $b_{2k+1} = 2^{-k}$ for $k = 0,1,2,\ldots$ and $b_{2k} = a_k$ for $k = 1,2,\ldots$ Find the sum of the series $\sum_{n=1}^{\infty} b_n$.

(a) s + 1 (b) s+ 2 (c) 2s (d) s-1 (e) s-2

35. Let $y = \ln(\ln(\ln x^2))$. Find $\frac{dy}{dx}$.

(a) $\frac{1}{\ln(\ln x^2)}$ (b) $\frac{1}{x \ln x \ln(\ln x^2)}$ (c) $\frac{1}{2x \ln x (\ln x^2)}$

(d) $\frac{1}{2x \ln(\ln x^2)}$ (e) $\frac{1}{2 \ln x \ \ln(\ln x^2)}$.

36. Find the maximum value of y = sin x (1+ cosx)

(a) $\frac{\sqrt{3}}{4}$ (b) $\frac{3\sqrt{3}}{4}$ (c) $\frac{3\sqrt{3}}{2}$ (d) $\frac{3}{4}$ (e) $\frac{3}{2}$

37. The solution set of the inequality $\frac{2x + 2}{x - 5} \leq 3$ is

(a) $(-\infty, 5) \cup (17, \infty)$ (b) $(-\infty, 5] \cup (17, \infty)$ (c) $(-\infty, 5) \cup [17, \infty)$

(d) $(-\infty, 5] \cup [17, \infty)$ (e) $(5, 17]$

38. Let $f(x) = \sqrt{x-7}$, $g(x) = x^2 + 3$. The domain of f o g is

(a) $[7, \infty)$ (b) the set of all real numbers (c) $(-\infty, -2] \cup [2, \infty)$ (d) $[4, \infty)$

(e) $(7, \infty)$

39. Find the rectangular coordinates of the center of the circle given in terms of the polar coordinates $r = 6 \cos \theta$.

(a) (0,3) (b) (3,0) (c) (0,0) (d) (0,6) (e) (6,0)

40. If $\tan^{-1}(\frac{-3}{5}) = \theta$, then $\cos 2\theta$ equals

(a) $\frac{-8}{17}$ (b) $\frac{8}{17}$ (c) $\frac{-8}{25}$ (d) $\frac{8}{25}$ (e) $\frac{41}{9}$