With the usual assumption that $u(x, t) = X(x)T(t)$, separating variables in (1) gives

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

so that

$$X'' + \lambda X = 0 \tag{4}$$

$$T'' + a^2 \lambda T = 0. \tag{5}$$

As in the preceding section, the boundary conditions (2) translate into $X(0) = 0$ and $X(L) = 0$. Equation (4) along with these boundary conditions is the regular Sturm-Liouville problem

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0. \tag{6}$$

Of the usual three possibilities for the parameter, $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$, only the last choice leads to nontrivial solutions. Corresponding to $\lambda = \alpha^2$, $\alpha > 0$, the general solution of (4) is

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x.$$

$X(0) = 0$ and $X(L) = 0$ indicate that $c_1 = 0$ and $c_2 \sin \alpha L = 0$. The last equation again implies that $\alpha L = n\pi$ or $\alpha = n\pi/L$. The eigenvalues and corresponding eigenfunctions of (6) are $\lambda_n = n^2\pi^2/L^2$ and $X(x) = c_2 \sin \dfrac{n\pi}{L} x$, $n = 1, 2, 3, \ldots$. The general solution of the second-order equation (5) is then

$$T(t) = c_3 \cos \frac{n\pi a}{L} t + c_4 \sin \frac{n\pi a}{L} t.$$

By rewriting c_2c_3 as A_n and c_2c_4 as B_n, solutions that satisfy both the wave equation (1) and boundary conditions (2) are

$$u_n = \left(A_n \cos \frac{n\pi a}{L} t + B_n \sin \frac{n\pi a}{L} t\right) \sin \frac{n\pi}{L} x \tag{7}$$

and

$$u(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi a}{L} t + B_n \sin \frac{n\pi a}{L} t\right) \sin \frac{n\pi}{L} x. \tag{8}$$

Setting $t = 0$ in (8) and using the initial condition $u(x, 0) = f(x)$ gives

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x.$$

Since the last series is a half-range expansion for f in a sine series, we can write $A_n = b_n$:

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx. \tag{9}$$

To determine B_n, we differentiate (8) with respect to t and then set $t = 0$:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-A_n \frac{n\pi a}{L} \sin \frac{n\pi a}{L} t + B_n \frac{n\pi a}{L} \cos \frac{n\pi a}{L} t\right) \sin \frac{n\pi}{L} x$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = \sum_{n=1}^{\infty} \left(B_n \frac{n\pi a}{L}\right) \sin \frac{n\pi}{L} x.$$

For this last series to be the half-range sine expansion of the initial velocity g on the interval, the *total* coefficient $B_n n\pi a/L$ must be given by the form b_n in (5), that is,

$$B_n \frac{n\pi a}{L} = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

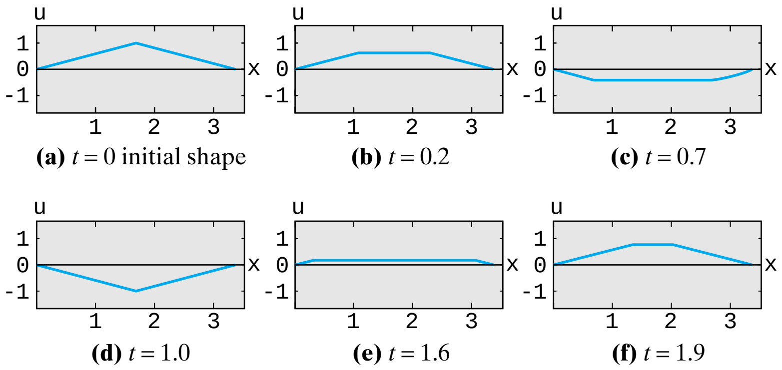
from which we obtain

$$B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx. \tag{10}$$

The solution of the boundary-value problem (1)–(3) consists of the series (8) with coefficients A_n and B_n defined by (9) and (10), respectively.

We note that when the string is released from *rest*, then $g(x) = 0$ for every x in the interval $0 \le x \le L$, and consequently, $B_n = 0$.

A special case of the boundary-value problem in (1)–(3) is the model of the **plucked string.** We can see the motion of the string by plotting the solution or displacement $u(x, t)$ for increasing values of time t and using the animation feature of a CAS. Some frames of a "movie" generated in this manner are given in Figure 12.8; the initial shape of the string is given in Figure 12.8(a). You are asked to emulate the results given in the figure plotting a sequence of partial sums of (8).



(a) $t = 0$ initial shape **(b)** $t = 0.2$ **(c)** $t = 0.7$

(d) $t = 1.0$ **(e)** $t = 1.6$ **(f)** $t = 1.9$

FIGURE 12.8 Frames of a CAS "movie"

Recall from the derivation of the one-dimensional wave equation in Section 12.2 that the constant a appearing in the solution of the boundary-value problem in (1), (2), and (3) is given by $\sqrt{T/\rho}$, where ρ is mass per unit length and T is the magnitude of the tension in the string. When T is large enough, the vibrating string produces a musical sound. This sound is the result of standing waves. The solution (8) is a superposition of product solutions called **standing waves** or **normal modes:**

$$u(x, t) = u_1(x, t) + u_2(x, t) + u_3(x, t) + \cdots.$$

In view of (6) and (7) , the product solutions (7) can be written as

$$u_n(x, t) = C_n \sin\left(\frac{n\pi a}{L}t + \phi_n\right) \sin\frac{n\pi}{L}x, \tag{11}$$

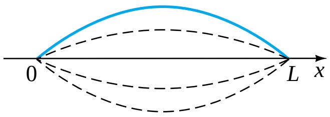
where $C_n = \sqrt{A_n^2 + B_n^2}$ and ϕ_n is defined by $\sin\phi_n = A_n/C_n$ and $\cos\phi_n = B_n/C_n$. For $n = 1, 2, 3, \ldots$ the standing waves are essentially the graphs of $\sin(n\pi x/L)$, with a time-varying amplitude given by

$$C_n \sin\left(\frac{n\pi a}{L}t + \phi_n\right).$$

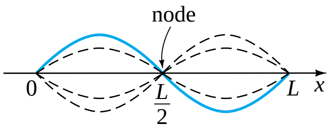
Alternatively, we see from (11) that at a fixed value of x each product function $u_n(x, t)$ represents simple harmonic motion with amplitude $C_n|\sin(n\pi x/L)|$ and frequency $f_n = na/2L$. In other words, each point on a standing wave vibrates with a different amplitude but with the same frequency. When $n = 1$,

$$u_1(x, t) = C_1 \sin\left(\frac{\pi a}{L}t + \phi_1\right) \sin\frac{\pi}{L}x$$

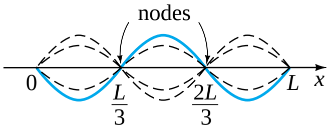
is called the **first standing wave,** the **first normal mode,** or the **fundamental mode of vibration.** The first three standing waves, or normal modes, are shown in Figure 12.9. The dashed graphs represent the standing waves at various values of time. The points in the interval $(0, L)$, for which $\sin(n\pi/L)x = 0$, correspond to points on a standing wave where there is no motion. These points are called **nodes.** For example, in Figures 12.9(b) and 12.9(c) we see that the second standing wave has one node at $L/2$ and the third standing wave has two nodes at $L/3$ and $2L/3$. In general, the nth normal mode of vibration has $n - 1$ nodes.



(a) first standing wave



(b) second standing wave



(c) third standing wave

FIGURE 12.9 First three standing waves

The frequency

$$f_1 = \frac{a}{2L} = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

of the first normal mode is called the **fundamental frequency** or **first harmonic** and is directly related to the pitch produced by a stringed instrument. It is apparent that the greater the tension on the string, the higher the pitch of the sound. The frequencies f_n of the other normal modes, which are integer multiples of the fundamental frequency, are called **overtones.** The second harmonic is the first overtone, and so on.