Final Examination Spring 2004

Name: Math 3130 100 points Score:

Show all steps in each problem to earn full credit. Each question carries 10 points.

1. a. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on the interval $[a, b]$. Show that

$$\|\phi_m(x)+\phi_n(x)\|^2 = \|\phi_m(x)\|^2 + \|\phi_n(x)\|^2, \quad m \neq n.$$

b. Find the norm of each function in the given orthogonal set on the interval [0, p].

$$\left\{\sin\frac{n\pi}{p}x\right\}, n = 1,2,3,...$$

2. a. Find the Fourier series of f on the indicated interval .

$$f(x) = x + \pi, \qquad -\pi < x < \pi$$

b. Use the result in part (a) to show that at $x = \frac{\pi}{2}$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

3. a. Show that if f

i. is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

ii. is odd, then $\int_{-a}^{a} f(x)dx = 0$

Expand each function in an appropriate cosine or sine series.

a.

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$$

c. $f(x) = |x|, -\pi < x < \pi$

4. a. Solve the boundary – value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0.$$

b. Show that for the eigenfunctions in part (a)

$$\left\| \sin\sqrt{\lambda_n}\, x \right\|^2 = \tfrac{1}{2}\left[1 + \cos^2\sqrt{\lambda_n}\right]$$

5. The Laguerre's differential equation $xy'' + (1-x)y' + ny = 0, \quad n = 0,1,2,...$ has polynomial solutions $L_n(x)$.

a. Put the equation in self – adjoint form .

b. Give an orthogonality relation.

6. Classify the given partial differential equations as hyperbolic, parabolic, or elliptic.

a. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

b. $3\dfrac{\partial^2 u}{\partial x^2} + 5\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

c. $\dfrac{\partial^2 u}{\partial x^2} + 6\dfrac{\partial^2 u}{\partial x \partial y} + 9\dfrac{\partial^2 u}{\partial y^2} = 0$

d. $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

7. Use separation of variables to find product solutions of

$$\frac{\partial^2 u}{\partial x \partial y} = u$$

8. Solve the wave equation

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L,\ t > 0$$

$$u(0,t) = 0, \quad u(\pi,t) = 0$$

$$u(x,0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin x$$

9. The solution of the heat equation

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}, \quad 0<x<L, \quad t>0$$
$$u(0,t)=0, \quad u(L,t)=0, \quad t>0$$
$$u(x,0)=f(x), \quad 0<x<L$$

is given by

$$u(x,t)=\frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx\right)e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t}\sin n x$$

Find $u(x,t)$ in the special case when $u(x,0)=100$, $L=\pi$, and $k=1$.

10. Solve the given Laplace's equation for the rectangular plate.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$$

$$u(0, y) = 0, \qquad u(a, y) = 0$$

$$u(x, 0) = f(x), \qquad u(x, b) = g(x)$$