Sections 1.2 and 1.3: Limits

Introduction

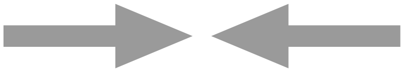
The thing that most separates the calculus from algebra is the concept of limit. Limits allow us to do remarkable things --- things that are impossible to do without them.

Consider the function $f(x) = \dfrac{x^2 + x - 2}{x - 1}$. We can substitute many different values in for x. But one value we cannot substitute in for x is 1. The function is undefined for $x = 1$.

Very often mathematicians need to know how a function behaves very near points where it is undefined. For instance, we might like to know how the function values (also known as y values) behave as we let x get very close to 1.

Numerical Method

For instance, we can consider the values of $f(x)$ for values of x that we choose closer and closer to 1. Let's make a table where we let x approach 1 from *both sides*:



x	0	0.9	0.99	1	1.01	1.1	2
$f(x)$				?			

Do the functional values appear to be getting closer to any particular value as x approaches 1?

It appears that when we chose x closer to 1 that $f(x)$ got closer to 3.

This method of evaluating a limit is known as a numerical method, and it is not always reliable. For instance, how do we know that when x gets even closer to 1 (say, 0.99999999999999999997 or 1.0000000000000000000000000000002), that the functional values don't start heading toward -41 or some other unexpected value? Such things *can* happen with certain functions! When we say "close," just how close do we mean? We have been somewhat vague so far.

We will eliminate some of that vagueness in the following definition:

Definition *(non-rigorous version)***:**
Let f be a function that is defined on some open interval containing c, except possibly at c itself. If there is a number L such that $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to c, but not equal to c, then we will say :

"the limit of $f(x)$ as x approaches c is L"

and write:

$$\lim_{x \to c} f(x) = L.$$

For instance, in the example we are looking at, it appears that the following may be true:

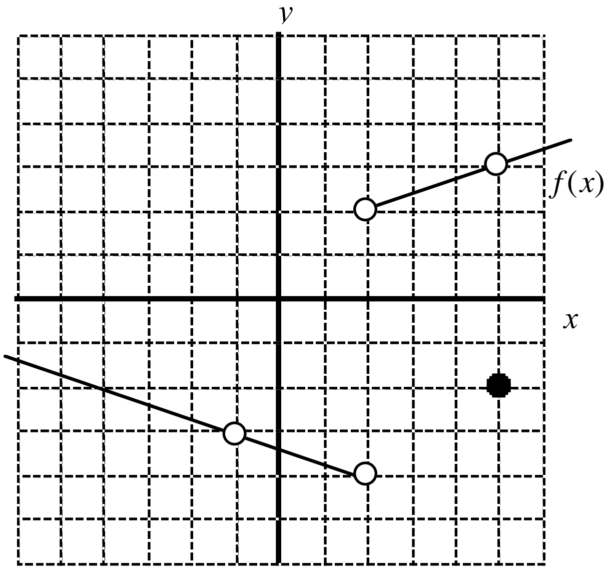
$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = 3.$$

Graphical Method

An evaluation method for limits that is somewhat similar to the numerical method -- at least in the idea behind the method -- is the <u>graphical</u> method. In this method, one examines the graph of the given function near the stated value of c to determine whether the functional values of $f(x)$ (the y values).

Consider the following limits:

Examples:
Let f be the function graphed below and evaluate the given limits if possible.



1. $\lim\limits_{x \to -4} f(x)$

2. $\lim\limits_{x \to -1} f(x)$

3. $\lim\limits_{x \to 2} f(x)$

4. $\lim\limits_{x \to 5} f(x)$

Analytical Method

The most reliable method for evaluation limits is the analytic method.

Consider our first example: $\lim\limits_{x \to 1} \dfrac{x^2 + x - 2}{x - 1}$.

Again, we let $f(x) = \dfrac{x^2 + x - 2}{x - 1}$. Then

$$f(x) = \frac{(x+2)(x-1)}{x-1}.$$

How does the function f compare to the function g defined by $g(x) = x + 2$?

Since we specifically said that we do not let x go all the way to the given value of c, we may write the following:

$$\begin{aligned}\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} &= \lim_{x \to 1} \frac{(x+2)(x-1)}{x-1} \\ &= \lim_{x \to 1} (x+2) \\ &= 1 + 2 = 3.\end{aligned}$$

Note that we used the limit notation in writing out the steps of our solution up until we substituted the value of c. At that point, we dropped the limit notation.

Example: $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x^2 - 4x}$

Note that if we substituted the given values of c into functions in the preceding examples, we would get $\frac{0}{0}$. Also note that we got different values for the two limits: the form $\frac{0}{0}$ tells us nothing about the eventual answer. For this reason we call the form $\frac{0}{0}$ an <u>indeterminate</u> <u>form</u>.

Example: $\lim\limits_{x \to 2} \dfrac{\frac{1}{x} - \frac{1}{2}}{x - 2}$

Example: $\lim\limits_{x \to 3} \dfrac{\sqrt{x+13} - 4}{x - 3}$

The Rigorous Definition of Limit

Recall our non-rigorous definition of limit:

Let f be a function that is defined on some open interval containing c, except possibly at c itself. If there is a number L such that $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to c, but not equal to c, then we will say :

“the limit of $f(x)$ as x approaches c is L”

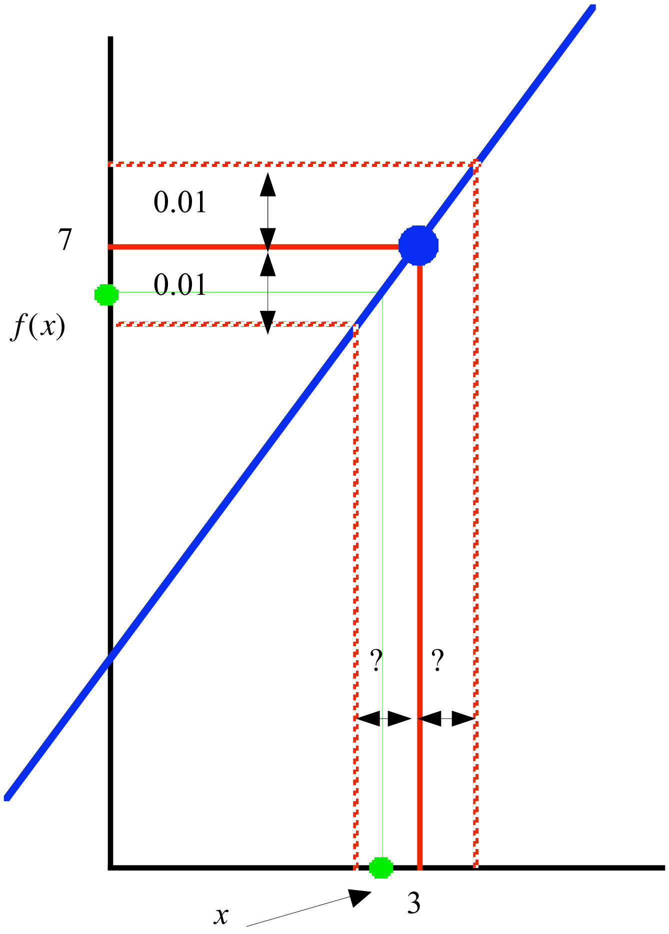
and write:

$$\lim_{x \to c} f(x) = L.$$

What does it mean that “$f(x)$ can be made arbitrarily close to L?”

Consider the example: $\lim\limits_{x \to 3} (2x+1)$. Since the limit is 7, we should be able to make $2x+1$ as close to 7 as we want by making x sufficiently close to 3.

Suppose we wish $2x+1$ to be within 0.01 of 7. How close must x be to 3 in order to guarantee this?



In this case, we want

$$6.99 < 2x + 1 < 7.01 .$$

We subtract 1:

$$5.99 < 2x < 6.01$$

and divide by 2:

$$2.995 < x < 3.005 .$$

Then x must be within 0.005 of 3.

Another way to solve the above problem is with absolute values. Recall that the distance between two real numbers a and b is given by

$$|a - b|.$$

We want the distance between $2x + 1$ and 7 to be less than 0.01, and we need to determine how small the x must be within 0.005 of 3.

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So we start with

$$|(2x + 1) - 7| < 0.01$$

and will finish with

$$|x - 3| < ?,$$

where we must replace the "?" with some number.

Then

$$\begin{aligned} |(2x+1)-7| &< 0.01 \\ |2x-6| &< 0.01 \\ |2(x-3)| &< 0.01 \\ |2|\,|x-3| &< 0.01 \\ 2|x-3| &< 0.01 \\ |x-3| &< \frac{0.01}{2} \\ |x-3| &< 0.005. \end{aligned}$$

Then x must be within 0.005 of 3, as we found before.

Recall the non-rigorous definition of limit again:

If there is a number L such that $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to c, but not equal to c, then we will say :

"the limit of $f(x)$ as x approaches c is L"

and write:

$$\lim_{x \to c} f(x) = L.$$

In a *rigorous* definition of limit, we will want to restate the ideas of the non-rigorous definition.

First,

"there is a number L such that $f(x)$ can be made arbitrarily close to L"

can be written as

"there is a number L such that the distance between $f(x)$ and L can be made arbitrarily small"

or, more mathematically,

"there is a number L such that $|f(x) - L|$ can be made arbitrarily small."

Now, what do we mean by "arbitrarily small?" Mathematicians interpret the phrase in this case as

"smaller than any arbitrary positive number ε."

So

"there is a number L such that $|f(x) - L|$ can be made arbitrarily small"

becomes

"there is a number L such that $|f(x) - L| < \varepsilon$ for any arbitrary positive number ε."

Now

"x is close to c"

means

"the distance between x and c is small"

or

"$|x - c|$ is small."

As before "small" will mean smaller than some positive number δ, so we will write

$$|x - c| < \delta.$$

We also have to include the provision that x is not equal to c. We can accomplish this by insisting that

$$0 < |x - c|.$$

We then arrive at the inequality

$$0 < |x - c| < \delta.$$

So here is the rigorous definition of limit at last:

Definition:
Let f be a function that is defined on some open interval containing c, except possibly at c itself. If there is a number L such that for every positive number ε, there is a positive number δ so that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$, then we will say :

"the limit of $f(x)$ as x approaches c is L"

and write:

$$\lim_{x \to c} f(x) = L.$$